Stochastic distortion and stochastic distorted copula

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Abstract

Motivated by the wide applications of distortion function and copulas in insurance and finance, this paper generalizes the notion of deterministic distortion function to a stochastic distortion function, i.e., a random process, and employs the defined stochastic distortion function to construct a so-called stochastic distorted copula. One method for constructing stochastic distortions is provided with a focus on using time-change processes. After giving some families of stochastic distorted copulas, the stochastic distorted copula is applied to a portfolio credit risk model with a numeric study to show the advantage of using stochastic distorted copulas over conventional Gaussian copula and double t copula in terms of fitting accuracy and catching tail dependence.

Keywords: Stochastic distortion, Stochastic distorted copula, Time-change process, Portfolio credit risk model.

1 Introduction

A function $D(u), u \in [0,1]$ is called a distortion function if it is non-decreasing with $D(0) = 0$ and $D(1) = 1$. Distortion function is also called weighting function or probability distortion in economic and behavioral studies, and it is one of the key elements of Kahneman and Tversky's Nobel-prize-winning theory, i.e., Prospect Theory and Cumulative Prospect Theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992). In Prospect Theory and Cumulative Prospect Theory, an inverse s-shaped distortion function is applied to enlarge the low probability and dismiss the large probability so as to reflect the human subjective probability. Distortion function was also applied in an experimental design on Cumulative Prospect Theory in Harrison and Swarthout (2016). With the advent of Cumulative Prospect Theory, distortion function has been used in the portfolio selection and behavioral related study. For example, Aït-Sahalia and Brandt (2001), Berkelaar et al. (2004), Jin and Zhou (2011) and Carassus and Rasonyi (2015) studied the portfolio selection problem under the utility framework by combining with distortion function; Cohen and Jaffray (1988), Bleichrodt and Pinto (2000) and Bruhin et al. (2010) applied probability distortion to study behavioral risk-taking and decision-making; Trepe et al. (2005), Zhang and Maloney (2012) and Stauffer et al. (2015) employed distortion function to the study of neurosceince.

It has been a long history for applying distortion function in risk measure. For example, Yaari (1987) formally applied the distortion function in dual theory of choice under risk; Wang (1996) defined Wang's premium principle by using distortion function and introduced the distortion risk measure, which covers some well-known risk measures such as Value-at-Risk and expected shorfall. The distortion risk measures

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are perspective risk measures where entity's attitude toward risk could be reflected by selecting appropriate distortion functions (Dhaene et al., 2006). The distortion risk measures have been applied for computing insurance premiums (Landsman and Sherris, 2001), capital requirement (Hurlimann, 2004), capital allocation (Tsanakas, 2004) and investment portfolio optimization (Sered et al., 2010).

Another important application of distortion function is to construct copula functions. A copula function is a multi-dimensional distribution function with all uniform $[0,1]$ marginal distributions. As an important method for modeling dependence between risks, copulas have been used in credit risk modeling (Laurent and Gregory, 2005; Burtschel and Gregory, 2005; Burtschell et al., 2012), actuarial science (Lindskog and McNeil, 2001; Deelstra et al., 2011) and risk management (Dhaene et al., 2006; Li et al., 2014). See Joe (1997) and Nelsen (2006) for a standard introduction to copulas and McNeil et al. (2015) for an overview of applications in risk management. Given the popularity of copula models, constructing copulas by using distortion functions has received attention too. For example, Genest and Rivest (2001), Klement et al. (2005), Durante and Sempi (2005) and Durante et al. (2010) considered a distortion transformation of a bivariate copula by distorting each component of the copula and inverting them; Morillas (2005) extended this idea to multivariate copulas by applying absolutely monotonic distortion functions; Li et al. (2014) introduced Distorted Mix Method to construct copula function by combining distortion functions with the convex sum method, which leads to modeling copula function's central part and tail parts separately.

This paper first extends the notion of distortion function to a stochastic distortion, which is a random process with the same properties as a distortion function. One method for constructing a stochastic distortion is provided by focusing on time-change processes. Since every semi-martingale is equivalent to a time-change Brownian motion (Monroe, 1978), the time-change method provides a highly flexible way to build more preferable financial models starting from some basic models, such as stochastic volatility models (Carr and Wu, 2004; Li and Linetsky, 2014), credit risk models (Gordy and Szerszen, 2015; Costin et al., 2014; Mendoza and Linetsky, 2016) and equilibrium pricing models (Shaliastovich and Tauchen, 2008). In this paper, three important types of time-change processes, Lévy subordinator, additive subordinator and absolutely continuous time-change process (Mendoza and Linetsky, 2016), are employed to construct stochastic distortions.

Secondly we use the defined stochastic distortion functions to construct copulas. The new copula function is constructed from an original copula by using stochastic distortions to change each component of it, which is called stochastic distorted copula. The stochastic distorted copula combines the information of the copula function with the multivariate stochastic distortion function. Some examples of stochastic distorted copulas are presented to demonstrate the usefulness and flexibility of the proposed method.

Finally we apply one type of stochastic distorted copula constructed from a linear factor model in Mendoza and Linetsky (2016) to the multiname credit risk model. The copula function of this type is similar to the commonly used factor Gaussian copula with the combination of systematic factor and idiosyncratic factors, and it can incorporate the tail dependence and capture the default clustering simultaneously. In comparison with Gaussian copula and double t copula, a numerical study is conducted to show that the proposed new model gives more desirable results in calibrating CDO tranches' price.

The remainder of the paper is organized as follows. Section 2 defines the stochastic distortion, and provides some families of stochastic distortions via transforming some time-change processes. Section 3 uses the defined stochastic distortions to construct a new copula named as stochastic distorted copula, and studies the density function and tail dependence of the stochastic distorted copula. Some classes of stochastic distorted copulas are given in Section 4. Section 5 is an application of stochastic distorted copulas to a portfolio credit risk model on fitting CDO market price. Conclusions are summarized in Section 6. Some proofs are put in the appendix.

2 Stochastic distortion

2.1 Definition of stochastic distortion

A distortion function $D(u), u \in [0,1]$ is a non-decreasing function from $[0,1]$ to $[0,1]$ with $D(0) = 0$ and $D(1) = 1$. Note that the distortion function is deterministic. In the next, we will introduce a notion named as stochastic distortion, which generalizes the concept of deterministic distortion function to a random process.

Definition 2.1. (Stochastic distortion) A stochastic process $X(u)$, $u \in [0,1]$ is called a stochastic distortion if $X(u)$, $u \in [0,1]$ is a non-decreasing process with $X(0) = 0$ and $X(1) = 1$ almost surely.

If $X(u), u \in [0,1]$ is a stochastic distortion, then it is easy to see that the process

$$\widehat{X}(u) = 1 - X(1-u), u \in [0,1]$$

is a stochastic distortion too, which is called the dual stochastic distortion of $X(u)$. Since the stochastic distortion $X(u), u \in [0,1]$ is a non-decreasing process, it allows us to introduce its inverse process

$$X^{-1}(u) = \inf\{v : X(v) \geq u\}, u \in [0,1],$$

which is a non-decreasing process as well. Throughout for each stochastic distortion $X(u), u \in [0,1]$, we define $X(u) = 1, u > 1$ and $X(u) = 0, u < 0$.

The following example gives a stochastic distortion constructed from a Poisson process.

Example 2.1. Suppose that $N_t, t \geq 0$ is a Poisson process with intensity $\lambda > 0$. The Laplace transform of N_t can be expressed as

$$E\left[e^{\theta N_t}\right] = \exp\left(t\psi(\theta)\right) \quad \text{for any} \quad \theta \leq 0,$$

where

$$\psi(\theta) = \lambda\left(e^{\theta} - 1\right).$$

For a fixed $\theta < 0$, we define the stochastic process $X_\theta(u), u \in [0,1]$ as

$$X_\theta(u) = 1 - \exp\left(\theta N_{\omega_\theta(1-u)}\right), u \in (0,1), \quad \text{and} \quad X_\theta(0) = \lim_{u \to 0^+} X_\theta(u), \quad X_\theta(1) = \lim_{u \to 1^-} X_\theta(u),$$

where $\omega_\theta(u) = \ln(u)/\psi(\theta)$. Since the Poisson process is a non-decreasing process with $P(N_0 = 0) = P(N_\infty = \infty) = 1$, we claim that X_θ is a non-decreasing process with $X_\theta(0) = 0$ and $X_\theta(1) = 1$ almost surely. That is, for each $\theta < 0$ the process X_θ is a stochastic distortion. Moreover, for $u \in (0,1)$,

$$E[X_\theta(u)] = E\left[1 - \exp\left(\theta N_{\omega_\theta(u)}\right)\right] = 1 - \exp\left(\psi(\theta)\,\omega_\theta(u)\right) = u$$

and $E[X_\theta(0)] = 0, E[X_\theta(1)] = 1$. Hence for a fixed $\theta < 0$, the stochastic distortion X_θ satisfies $E[X_\theta(u)] = u, u \in [0,1]$.

Some properties of a stochastic distortion are summarized as follows. Its proof is put in the appendix.

Proposition 2.1. *Consider the stochastic distortion $X(u), u \in [0,1]$.*
(1) If $E[X(u)], u \in [0,1]$ is continuous, then the process $X(u), u \in [0,1]$ is stochastic continuous, i.e., given any $\varepsilon > 0$,

$$\lim_{h \to 0} P(|X(u+h) - X(u)| \geq \varepsilon) = 0;$$

(2) If $E[X(u)], u \in [0,1]$ *is continuous, then for* $a, b \in [0,1]$ *it holds that*

$$P\left(\left\{X^{-1}(a) \leq b\right\} \Delta \{X(b) \geq a\}\right) = 0, \tag{1}$$

where Δ *is the set operation of symmetric difference;*
(3) Assume U *is a uniform* $[0,1]$ *random variable and is independent of the stochastic distortion* $X(u), u \in [0,1]$*. Then* $X^{-1}(U)$ *is a uniform* $[0,1]$ *random variable if and only if* $E[X(u)] = u, u \in [0,1]$*;*
(4) Assume U *is a uniform* $[0,1]$ *random variable and is independent of the stochastic distortion* $X(u), u \in [0,1]$*. If* $E[X(u)] = u, u \in [0,1]$*, then* $X(U)$ *is larger than* U *in convex order, i.e., for any convex function* ϕ *we have*

$$E[\phi(X(U))] \geq E[\phi(U)].$$

2.2 Construction of stochastic distortions

Example 2.1 provides a basic idea for constructing stochastic distortions from a non-decreasing stochastic process. The next theorem presents a general approach for constructing a stochastic distortion.

Theorem 2.1. *Suppose that* $T_t, t \in [0,\infty)$ *is a nonnegative and non-decreasing stochastic process,* $\lim_{t\to 0^+} T_t = T_0 = 0, \lim_{t\to\infty} T_t = \infty$ *almost surely, and the function* $G(t), t \in [0,\infty)$ *is a continuous monotonic function with* $\inf_{t\in[0,\infty)} G(t) = 0, \sup_{t\in[0,\infty)} G(t) = 1$*. If* $g(t) = E[G(T_t)], t \in [0,\infty)$ *is strictly monotonic and* g^{-1} *denotes its inverse function, then for any continuous monotone function* $\alpha(u), u \in [0,1]$ *satisfying*

$$\lim_{u\to 0^+} \alpha(u) = g^{-1}(0), \lim_{u\to 1^-} \alpha(u) = g^{-1}(1),$$

the random process

$$X(u) = G\left(T_{\alpha(u)}\right), u \in (0,1), \quad \textit{and} \quad X(0) = \lim_{u\to 0^+} X(u), \quad X(1) = \lim_{u\to 1^-} X(u)$$

is a stochastic distortion. Specially, if $g(t), t \in [0,\infty)$ *is continuous and strictly monotonic, then for* $\alpha(u) = g^{-1}(u), u \in [0,1]$ *the stochastic distortion* $X(u), u \in [0,1]$ *satisfies that* $E[X(u)] = u, u \in [0,1]$*.*

Proof. Without loss of generality, we assume that $G(t), t \in [0,\infty)$ is non-decreasing. Thus the function $g(t) = E[G(T_t)]$, $t \in [0,\infty)$ is non-decreasing, and $g^{-1}(t), t \in [0,\infty)$ is well defined and it is non-decreasing. Then $g^{-1}(0) \leq g^{-1}(1)$ and $\alpha(u), u \in [0,1]$ is non-decreasing. From $\inf_{t\in[0,\infty)} G(t) = 0$ and $\sup_{t\in[0,\infty)} G(t) = 1$, we have $g(0) = 0$, $\lim_{t\to\infty} g(t) = 1$, which imply that $\lim_{u\to 0^+} \alpha(u) = g^{-1}(0) = 0$ and $\lim_{u\to 1^-} \alpha(u) = g^{-1}(1) = \infty$. Therefore $X(u), u \in [0,1]$ is a non-decreasing stochastic process with

$$X(0) = \lim_{u\to 0^+} G\left(T_{\alpha(u)}\right) = 0 \quad \text{and} \quad X(1) = \lim_{u\to 1^-} G\left(T_{\alpha(u)}\right) = 1 \quad \text{almost surely,}$$

i.e., $X(u), u \in [0,1]$ is a stochastic distortion. Specially, when $g(t), t \in [0,\infty)$ is strictly increasing and continuous, for $\alpha(u) = g^{-1}(u)$ we have

$$E[X(u)] = E\left[G\left(T_{g^{-1}(u)}\right)\right] = g\left(g^{-1}(u)\right) = u, u \in (0,1)$$

and $E[X(0)] = 0$, $E[X(1)] = 1$.

□

Theorem 2.1 shows that a stochastic distortion can be constructed easily and flexibly from a non-decreasing stochastic process. Example 2.1 is a special case in that we choose T_t as a Poisson process and $G(x) = 1 - e^{\theta x}, \theta < 0$. Since Laplace transform of a stochastic process has usually been well-studied in the literature, this paper mainly focuses on using the function $G(x) = 1 - e^{\theta x}, \theta < 0$ or $G(x) = e^{\theta x}, \theta < 0$ for constructing stochastic distortions via the Laplace transform of a stochastic process.

The following corollary is easily derived from Theorem 2.1.

Corollary 2.1. *Let $T_t, t \in [0,\infty)$ be a nonnegative and non-decreasing stochastic process, $\lim_{t\to 0^+} T_t = T_0 = 0, \lim_{t\to\infty} T_t = \infty$ almost surely, and its Laplace transform is given by*

$$E\left[e^{\theta T_t}\right] = e^{K_\theta(t)}, \theta < 0. \tag{2}$$

Then for fixed $\theta < 0$ and $\omega(u) = K_\theta^{-1}(\ln(u))$, the random process

$$X^{LT}(u) = \exp\left(\theta T_{\omega(u)}\right), u \in (0,1), \quad and \quad X^{LT}(0) = \lim_{u\to 0^+} X^{LT}(u), \quad X^{LT}(1) = \lim_{u\to 1^-} X^{LT}(u) \tag{3}$$

is a stochastic distortion satisfying $E\left[X^{LT}(u)\right] = u, u \in [0,1]$.

For comparing X^{LT} with its dual stochastic distortion $\widehat{X}^{LT}$, Table 1 reports their differences as u tends to zero.

Stochastic distortion and its dual stochastic distortion	$u \to 0^+$
$X^{LT}(u) = \exp\left(\theta T_{\omega(u)}\right), \omega(u) = K_\theta^{-1}(\ln(u))$	$\omega(u) \to \infty, T_{\omega(u)} \to \infty, X^{LT}(u) \to 0$
$\widehat{X}^{LT}(u) = 1 - \exp\left(\theta T_{\widehat{\omega}(u)}\right), \widehat{\omega}(u) = K_\theta^{-1}(\ln(1-u))$	$\widehat{\omega}(u) \to 0, T_{\widehat{\omega}(u)} \to 0, \widehat{X}^{LT}(u) \to 0$

Table 1: Comparison of stochastic distortion and its dual stochastic distortion.

Time-change process, especially the time-change Brownian motion, is a standard process for connecting diffusions and Brownian motions. Since Monroe (1978) showed that any semi-martingale is a time-change Brownian motion and it is known that most stochastic processes used in finance are semi-martingales, thus the time-change technique has become quite popular in the literature of finance. There are three important types of time-change processes, Lévy subordinator, additive subordinator and absolutely continuous time-change process (Mendoza and Linetsky, 2016). Based on these types of time-change processes, we will construct stochastic distortions by applying Theorem 2.1.

2.2.1 Laplace transform of Lévy subordinator

Lévy process is essentially a stochastic process with stationary and independent increments. Formally a Lévy process $\mathbf{L}_t$ on $\mathbb{R}^d$ is typically described by its triplet $(\mathbf{A}, \mathbf{N}, \gamma)$, such that its characteristic function can be expressed as

$$E\left[\exp\left(i\mathbf{z}^T \mathbf{L}_t\right)\right] = \exp\left(t\psi(\mathbf{z})\right), \mathbf{z} \in \mathbb{R}^d$$

with

$$\psi(\mathbf{z}) = -\frac{1}{2}\mathbf{z}^T \mathbf{A}\mathbf{z} + i\gamma^T \mathbf{z} + \int_{\mathbb{R}^d} \left(e^{i\mathbf{z}^T \mathbf{x}} - 1 - i\mathbf{z}^T \mathbf{x} \mathbf{1}_{||\mathbf{x}|| \le 1}\right) \mathbf{N}(d\mathbf{x}),$$

where $\mathbf{A}$,γ and $\mathbf{N}$ are called diffusion term, drift term and multi-dimensional Lévy measure, respectively. We can also express the above function as

$$\psi(\mathbf{z}) = -\frac{1}{2}\mathbf{z}^T \mathbf{A}\mathbf{z} + i\mathbf{b}^T \mathbf{z} + \int_{\mathbb{R}^d} \left(e^{i\mathbf{z}^T \mathbf{x}} - 1\right) \mathbf{N}(d\mathbf{x}),$$

where $\mathbf{b} = \gamma - \int_{||\mathbf{x}|| \le 1} \mathbf{x}\mathbf{N}(d\mathbf{x})$. If $\mathbf{A} = 0$, $\mathbf{b} \in \mathbb{R}_+^d$ and $\mathbf{N}$ is a σ-finite measure on $\mathbb{R}^d$ concentrated on $\mathbb{R}_+^d \setminus \{0\}$ such that $\int_{\mathbb{R}_+^d} (||\mathbf{x}|| \wedge 1) \mathbf{N}(d\mathbf{x}) < \infty$, then $\mathbf{L}_t$ is called a multivariate Lévy subordinator. See Cont and Tankov (2004) and Mendoza and Linetsky (2016) for details.

For one-dimensional Lévy subordinator $T_t, t \ge 0$, we have

$$K_\theta(t) = tl(\theta) \tag{4}$$

in (2) with

$$l(\theta) = b\theta + \int_0^\infty \left(e^{\theta x} - 1\right) \nu(dx), \tag{5}$$

where ν is a one-dimensional Lévy measure. Here $l(\theta)$ is called the Laplace exponent of the Lévy subordinator T_t. Obviously, $l(\theta)$ is an increasing and convex function with $l(\theta) \le l(0) = 0$ for $\theta \le 0$. In the case $\max(b, \nu((0,\infty])) > 0$, we have $\lim_{t\to\infty} E\left[e^{-T_t}\right] = 0$, which implies that $\lim_{t\to\infty} T_t = \infty$ almost surely. Hence X^{LT} in (3) is a stochastic distortion with $E\left[X^{LT}(u)\right] = u, u \in [0,1]$.

Example 2.2. For the one-dimensional Lévy subordinator $T_t, t \ge 0$, we assume $b = 0$ and focus on the Lévy measure ν with the additional restriction $E[T_t] = t$, i.e.,

$$\int_0^\infty x\nu(dx) = 1.$$

When the tail of the Lévy measure decays exponentially, i.e.,

$$\nu(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{1-\alpha}{\kappa}\right)^{1-\alpha} \frac{e^{-(1-\alpha)x/\kappa}}{x^{1+\alpha}} 1_{x>0}, \tag{6}$$

where κ is a positive constant and $0 \le \alpha < 1$, the corresponding Lévy subordinator is called as a tempered stable subordinator. The family of tempered stable subordinators includes two special types of processes, the Gamma process ($\alpha = 0$) and the inverse Gaussian process ($\alpha = 1/2$). The parameter α is the index of stability and it determines the relative importance of small jumps in the path of the process. The parameter $\kappa = Var(T_1)$ describes the randomness of the time-change process. When κ tends to zero, the effect of time-change perturbation tends to be vanished. By (5), the Laplace exponent of the tempered stable subordinator can be expressed as

$$l(\theta) = \begin{cases} \frac{1-\alpha}{\kappa\alpha}\left[1 - \left(1 - \frac{\kappa\theta}{1-\alpha}\right)^\alpha\right], & 0 < \alpha < 1, \\ -\frac{\ln(1-\kappa\theta)}{k}, & \alpha = 0. \end{cases} \tag{7}$$

Thus from (4) and (7), by applying (2) and (3), we can derive the expression for the stochastic distortion transformed from the tempered stable subordinator.

Remark 2.1. Let T_t be the first time required for $u + W_u$ to reach the value t, where W_u is a standard Brownian motion. Then it is known that T_t is an inverse Gaussian process with $E[T_t] = t$ and $Var(T_1) = 1$. In other words, the inverse Gaussian process can be seen as the first passage time of the drifted Brownian motion. Thus Example 2.2 shows that we can apply Brownian motion to construct stochastic distortions as well.

2.2.2 Laplace transform of the additive subordinator

Additive subordinator is an extension of Lévy subordinator without assuming stationary increment. Thus it is generally time inhomogeneous. For an additive subordinator $T_t, t \ge 0$ and $\theta \le 0$, it satisfies that

$$E\left[e^{\theta T_t}\right] = e^{\int_0^t l(\theta, y)dy}, t \ge 0$$

with

$$l(\theta,y)=\theta b(y)+\int_0^{\infty}\left(e^{\theta x}-1\right)\nu(y,dx),$$

where $b(y)\geq 0$ and $\int_0^{\infty}(x\wedge 1)\,\nu(y,dx)<\infty$. Thus we have

$$K_\theta(t)=\int_0^t l(\theta,y)\,dy$$

in (2) and (3). It is easy to verify that $T_0=0$. On the other hand, if $\int_0^{\infty} l(\theta,y)\,dy=-\infty$, we have $\lim_{t\to\infty}E\left[e^{\theta T_t}\right]=0$, which implies that $\lim_{t\to\infty}T_t=\infty$ almost surely.

2.2.3 Laplace transform of the absolutely continuous time-change process

The stochastic distortions transformed by Lévy subordinators and additive subordinators are jump processes. In the following, we will introduce stochastic distortions transformed from absolutely continuous time-change processes.

Let V_t be a nonnegative activity rate process with the initial value V_0. The cumulative activity rate $T_t=\int_0^t V_s ds, t>0$ is an absolutely continuous process. The choice of an activity rate process V_t is flexible, such as the square of any non-vanished stochastic process. For instance, if the activity rate process V_t is chosen as a CIR process

$$dV_t=b(a-V_t)\,dt+\sigma\sqrt{V_t}dW_t,$$

where a,b,σ are positive constants and W_t is the standard Brownian motion, then we have

$$K_\theta(t)=\alpha(\theta,t)+\beta(\theta,t)V_0,\theta\leq 0$$

in (2) and (3), where $\alpha(\theta,t)$ and $\beta(\theta,t)$ are the exponential affine coefficients of the CIR process. See Duffie et al. (2000) for the mathematical expression of $\alpha(\theta,t)$ and $\beta(\theta,t)$. By Duffie et al. (2000), we have

$$\lim_{t\to\infty}E\left[\exp\left(-\int_0^t V_s ds\right)\right]=0,$$

which implies $T_t=\int_0^t V_s ds\to\infty$ almost surely as $t\to\infty$.

3 Stochastic distorted copula

In this section, we will show how stochastic distortion can be employed to construct new copulas.

3.1 Definition of stochastic distorted copula

Given n-dimensional copula function B and multivariate stochastic distortion

$$\mathbf{X}(u)=(X_1(u),\ldots,X_n(u)),$$

for $u_i\in[0,1], i=1,\ldots,n$ we define

$$C^{\mathbf{X}|B}(u_1,\ldots,u_n)=E\left[B(X_1(u_1),X_2(u_2),\ldots,X_n(u_n))\right]. \tag{8}$$

The above definition combines the information of the copula function B with the multivariate stochastic distortion $(X_1(u),\ldots,X_n(u))$. An interesting question is when the above formula defines a new copula.

To begin with, we recall the definition of positive quadrant dependent (PQD) order, which is the most common stochastic order of positive dependence in the literatures. Let $\mathbf{U} = (U_1, U_2, \ldots, U_n)$ be a random vector with distribution F and survival function $\widehat{F}$, and $\mathbf{V} = (V_1, V_2, \ldots, V_n)$ be a random vector with distribution G and survival function $\widehat{G}$. If $F(\mathbf{u}) \leq G(\mathbf{u})$ and $\widehat{F}(\mathbf{u}) \leq \widehat{G}(\mathbf{u})$ for all $\mathbf{u}$, then we say $\mathbf{U}$ is smaller than $\mathbf{V}$ in the PQD order, denoted by $\mathbf{U} \leq_{PQD} \mathbf{V}$ or $F \leq_{PQD} G$. By selecting $\mathbf{u} = (\infty, \ldots, \infty, u_i, \infty, \ldots, \infty)$, we can see that only the random vectors with the common univariate marginal distributions can be compared in the PQD order. For more details on PQD order, see Chapter 9 of Shaked and Shanthikumar (2007).

In the family of copula functions, we denote the Fréchet upper bound by $C_+(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n)$, $u_1, \ldots, u_n \in [0,1]$, the product copula by $\Pi(u_1, \ldots, u_n) = \prod_{i=1}^n u_i, u_1, \ldots, u_n \in [0,1]$, and the Fréchet lower bound by $C_-(u_1, \ldots, u_n) = \max(u_1 + \ldots + u_n - n + 1, 0), u_1, \ldots, u_n \in [0,1]$. It is known that the Fréchet lower bound $C_-(u_1, \ldots, u_n)$ is a copula function only when $n = 2$. Here we also call $C_+(u_1, \ldots, u_n)$ as the comonotonic copula, and $C_-(u_1, u_2)$ as the countermonotonic copula.

Theorem 3.1. *(Stochastic distorted copula) Suppose that for $i = 1, \ldots, n$, the stochastic distortion X_i satisfies $E[X_i(u)] = u, u \in [0,1]$ and B is a copula function.*
(1) The function $C^{\mathbf{X}|B}$ in (8) is a copula function. Moreover, if the random vector $(U_1, \ldots, U_n)$ has distribution B and $(U_1, \ldots, U_n)$ is independent of the stochastic distortions $X_i(u), u \in [0,1], i = 1, \ldots, n$, then the random vector $\left(X_1^{-1}(U_1), \ldots, X_n^{-1}(U_n)\right)$ has distribution $C^{\mathbf{X}|B}$.
(2) For copula functions B_1, B_2 satisfying that $B_1 \leq_{PQD} B_2$, we have

$$C^{\mathbf{X}|B_1} \leq_{PQD} C^{\mathbf{X}|B_2}. \tag{9}$$

(3) When $n = 2$, we have

$$C^{\mathbf{X}|C_-} \leq_{PQD} C^{\mathbf{X}|B} \leq_{PQD} C^{\mathbf{X}|C_+}, \quad \rho_{C^{\mathbf{X}|C_-}} \leq \rho_{C^{\mathbf{X}|B}} \leq \rho_{C^{\mathbf{X}|C_+}}, \tag{10}$$

where ρ_C represents the correlation index of copula function C, including linear correlation coefficient, Kendall's τ, Spearman's ρ, and Blomquist's q. Specially, if $X_1(u_1)$ and $X_2(u_2)$ are mutually independent, we have

$$C^{\mathbf{X}|C_-} \leq_{PQD} \Pi \leq_{PQD} C^{\mathbf{X}|C_+}, \quad \rho_{C^{\mathbf{X}|C_-}} \leq 0 \leq \rho_{C^{\mathbf{X}|C_+}}. \tag{11}$$

For the copula function defined in (8), we call it a stochastic distorted copula, and the copula function B is named as the base copula of the stochastic distorted copula. The stochastic distorted copula $C^{\mathbf{X}|B}$ can be regarded as changing each component of the copula function B by the stochastic distortions $X_1, \ldots, X_n$. Note that the stochastic distortions $X_1, \ldots, X_n$ may be correlated or independent.

It follows from Theorem 3.1 that the stochastic distortions keep the PQD order and the correlation index $\rho_{C^{\mathbf{X}|B}}$ is bounded by $\rho_{C^{\mathbf{X}|C_-}} \leq \rho_{C^{\mathbf{X}|B}} \leq \rho_{C^{\mathbf{X}|C_+}}$ in the two-dimensional case.

Another fundamental transformation-based method for constructing copula function is the convex sum. Combining the ideas of stochastic distortion and convex sum, we directly have the following corollary. Its proof is omitted.

Corollary 3.1. *Assume that the parameter $Z \in \Xi$ is a random variable and for each $z \in \Xi$ the function B_z is a copula. If for each fixed $i = 1, \ldots, n$, conditional on Z the random process X_i is a stochastic distortion with $E[X_i(u)|Z] = u, u \in [0,1]$, then*

$$C^{\mathbf{X},Z|B_{\mathbf{Z}}}(u_1, \ldots, u_n) \triangleq E\left[B_Z(X_1(u_1), \ldots, X_n(u_n))\right], u_1, \ldots, u_n \in [0,1]$$

is a copula function.

One example of the above stochastic distorted copula is constructed as follows. Let Z be an index variable with distribution $P(Z=j)=\alpha_j,\ j\geq 1$. Consider n-dimensional copula functions $B_j,\ j=1,2,\ldots,$ and different distortion functions $D_{i,j}, i,j\geq 1$ satisfying

$$\sum_j \alpha_j D_{i,j}(u)=u,\ u\in[0,1].$$

Put $X_i(u)=D_{i,Z}(u), u\in[0,1]$. Then $X_i, i=1,\ldots,n$, are stochastic distortions satisfying the condition in Corollary 3.1 with finite trajectories

$$P\left(X_i(u)=D_{i,j}(u), u\in[0,1]\right)=P(Z=j)=\alpha_j,\ i=1,\ldots,n,\ j\geq 1.$$

Therefore it follows from Corollary 3.1 that

$$C^{\mathbf{X},Z|B_{\mathbf{Z}}}(u_1,\ldots,u_n)=E[B_Z(D_{1,Z}(u_1),\ldots,D_{n,Z}(u_n)]=\sum_j \alpha_j B_j\left(D_{1,j}(u_1),\ldots,D_{n,j}(u_n)\right) \tag{12}$$

is a copula function, which is consistent with the Distorted Mix Method (DMM) in Li et al. (2014). In other words, the above method in Corollary 3.1 can be viewed as a generalization of the DMM method in Li et al. (2014).

The following proposition shows that the survival copula of a stochastic distorted copula also has the form of (8) with survival base copula and dual stochastic distortions.

Proposition 3.1. *Under the conditions of Theorem 3.1, $C^{\widehat{\mathbf{X}}|\widehat{B}}$ is the survival copula of $C^{\mathbf{X}|B}$, where $\widehat{B}$ is the survival copula of B and $\widehat{X}_i$ is the dual stochastic distortion of $X_i, i=1,\ldots,n$.*

Proof. Let $(U_1,\ldots,U_n)$ be a random vector with distribution B and assume that $(U_1,\ldots,U_n)$ is independent of the stochastic distortions $X_1,\ldots,X_n$. Then $(1-U_1,\ldots,1-U_n)$ has copula $\hat{B}$. By Theorem 3.1, we have

$$\begin{aligned}\widehat{C^{\mathbf{X}|B}}(u_1,\ldots,u_n) &= P\left(1-X_1^{-1}(U_1)\leq u_1,\ldots,1-X_n^{-1}(U_n)\leq u_n\right)\\ &= P\left(X_1^{-1}(U_1)\geq 1-u_1,\ldots,X_n^{-1}(U_n)\geq 1-u_n\right).\end{aligned}$$

Note that $X_i^{-1}(U_i)$ is Uniform [0,1] random variable, it follows from the above equation that

$$\begin{aligned}\widehat{C^{\mathbf{X}|B}}(u_1,\ldots,u_n) &= P\left(X_1^{-1}(U_1)>1-u_1,\ldots,X_n^{-1}(U_n)>1-u_n\right)\\ &= P\left(U_1>X_1(1-u_1),\ldots,U_n>X_n(1-u_n)\right).\end{aligned}$$

Since $\widehat{X}_i(u_i)=1-X_i(1-u_i)$, we further have

$$\begin{aligned}\widehat{C^{\mathbf{X}|B}}(u_1,\ldots,u_n) &= P\left(1-U_1<\widehat{X}_1(u_1),\ldots,1-U_n<\widehat{X}_n(u_n)\right)\\ &= P\left(1-U_1\leq\widehat{X}_1(u_1),\ldots,1-U_n\leq\widehat{X}_n(u_n)\right)\\ &= E\left[\widehat{B}\left(\widehat{X}_1(u_1),\ldots,\widehat{X}_n(u_n)\right)\right]\\ &= C^{\widehat{\mathbf{X}}|\widehat{B}}(u_1,\ldots,u_n),\quad u_1,\ldots,u_n\in[0,1],\end{aligned}$$

which completes the proof. □

Indeed some correlation models in credit risks can be derived by using a stochastic distorted copula. The following example illustrates such an application of stochastic distorted copula in credit portfolio.

Example 3.1. Suppose that each entity in the financial market has a business time reflecting the information rate. Let $\widetilde{\tau}_1, \widetilde{\tau}_2, \ldots, \widetilde{\tau}_n$ be the default times of n entities in business time and each $\widetilde{\tau}_i$ has a constant intensity $\widetilde{\lambda}_i$. We denote the calendar default times of the n entities by $\tau_1, \tau_2, \ldots, \tau_n$. For $i = 1, \ldots, n$, the business time $\widetilde{\tau}_i$ and the calendar time τ_i are connected by $\widetilde{\tau}_i = T^{(i)}_{\tau_i}$, where $T^{(i)}_t$ is a stochastic time-change process mapping calendar time to business time, and we also assume that $T^{(i)}_t$ is independent of $\tau_1, \tau_2, \ldots, \tau_n$. The copula function among $\widetilde{\tau}_1, \widetilde{\tau}_2, \ldots, \widetilde{\tau}_n$ is denoted by C. Then for $U_i = 1 - \exp\left(-\widetilde{\lambda}_i \widetilde{\tau}_i\right), i = 1, \ldots, n$, we have

$$C(u_1, \ldots, u_n) = P(U_1 \le u_1, \ldots, U_n \le u_n).$$

The survival function of $\widetilde{\tau}_i$ can be expressed as $S_i(t_i) = \exp\left(-\widetilde{\lambda}_i t_i\right)$.

Following the assumption in Gordy and Szerszen (2015), we suppose that $T^{(i)}_t$ is an inverse Gaussian process with mean t and variance $\kappa_i t$, i.e., its Laplace transform is $E\left[e^{\theta T^{(i)}_t}\right] = \exp(t\psi_i(\theta))$ with $\psi_i(\theta) = \frac{1}{\kappa_i}\left(1 - \sqrt{1 - 2\kappa_i\theta}\right), \theta \le 0$. Under the above setup, the joint distribution of $\tau_1, \tau_2, \ldots, \tau_n$ can be expressed as

$$\begin{aligned} P(\tau_1 \le t_1, \ldots, \tau_n \le t_n) &= P\left(\widetilde{\tau}_1 \le T^{(1)}_{t_1}, \ldots, \widetilde{\tau}_n \le T^{(n)}_{t_n}\right) \\ &= E\left[C\left(1 - \exp\left(-\widetilde{\lambda}_1 T^{(1)}_{t_1}\right), \ldots, 1 - \exp\left(-\widetilde{\lambda}_n T^{(n)}_{t_n}\right)\right)\right] \end{aligned}$$

with marginal distribution $P(\tau_i \le t_i) = 1 - E\left[\exp\left(-\widetilde{\lambda}_i T^{(i)}_{t_i}\right)\right] = 1 - \exp\left(t_i \psi_i\left(-\widetilde{\lambda}_i\right)\right)$. Thus the copula function among $\tau_1, \tau_2, \ldots, \tau_n$ can be expressed as

$$C^{tc}(u_1, \ldots, u_n) = E\left[C\left(1 - \exp\left(-\widetilde{\lambda}_1 T^{(1)}_{\Theta_1(1-u_1)}\right), \ldots, 1 - \exp\left(-\widetilde{\lambda}_n T^{(n)}_{\Theta_n(1-u_n)}\right)\right)\right] \quad (13)$$

with $\Theta_i(u_i) = \ln(u_i)/\psi_i\left(-\widetilde{\lambda}_i\right)$, where

$$X_i(u) = 1 - \exp\left(-\widetilde{\lambda}_i T^{(i)}_{\Theta_i(u)}\right), u \in (0,1), \quad \text{and} \quad X_i(0) = \lim_{u \to 0^+} X_i(u), \quad X_i(1) = \lim_{u \to 1^-} X_i(u), \quad (14)$$

which is a stochastic distortion with $E[X_i(u)] = u, i = 1, \ldots, n, u \in [0,1]$. Hence, it follows from Theorem 3.1 that the function C^{tc} in (13) is a stochastic distorted copula.

The stochastic distorted copula C^{tc} comes from the base copula C by adding the perturbation of the stochastic distortions $X_1, \ldots, X_n$. Note that κ_i equals to the variance of $T^{(i)}_1$, thus $T^{(i)}_t$ converges to t almost surely as κ_i tends to zero. That is, $C^{tc}(u_1, \ldots, u_n)$ converges to $C(u_1, \ldots, u_n)$ as $\kappa_i, i = 1, \ldots, n$ all tend to zero. Note that the dual stochastic distortion of the stochastic distortion (14) is a special type of the tempered stable subordinator distortion with $\alpha = 1/2, \theta = -\widetilde{\lambda}$ in (7).

3.2 Density function of the stochastic distorted copula

For the stochastic distorted copula introduced above, one may wonder how to calculate its density function. In the following, we show this can be done by simply using the transition operator for Markov processes and the infinitesimal generator for Feller processes.

To begin with, we first review the transition operator for Markov processes and the infinitesimal generator for Feller processes. The transition operator for a Markov process $X_t, t \ge 0$ is defined as

$$\mathscr{P}_t f(x) = E[f(X_t + x)].$$

Let $\mathscr{C}_0$ be the set of continuous functions vanishing at infinity. The Markov process X_t is called a Feller process if for any $f \in \mathscr{C}_0$ and fixed $t > 0$, the function $\mathscr{P}_t f(x), x \in (-\infty, \infty)$ belongs to $\mathscr{C}_0$ and $\lim_{t\to 0^+} \mathscr{P}_t f(x) = f(x), x \in (-\infty, \infty)$. A Feller process can be described by its infinitesimal generator $\mathscr{A}$ defined as

$$\mathscr{A} f(x) = \lim_{t\to 0^+} \frac{1}{t}\left(\mathscr{P}_t f(x) - f(x)\right) = \lim_{t\to 0^+} \frac{1}{t}\left(E\left[f\left(X_t + x\right)\right] - f(x)\right) \tag{15}$$

for $f \in \mathscr{D}(\mathscr{A})$. Here $\mathscr{D}(\mathscr{A})$ represents the domain of the infinitesimal generator of X_t, that is, the subset of $\mathscr{C}_0$ such that the right-hand side of (15) exists. As we know, a Lévy process is a Feller process. Below we will use the stochastic distorted copula constructed from the Lévy subordinator to show how the density function of a stochastic distorted copula can be computed.

Consider the Lévy subordinator distortions

$$X_i(u) = 1 - \exp\left(\theta_i T^{(i)}_{\widehat{\omega}_i(u)}\right), u \in (0,1), \quad \text{and} \quad X_i(0) = \lim_{u\to 0^+} X_i(u), \quad X_i(1) = \lim_{u\to 1^-} X_i(u)$$

with $\widehat{\omega}_i(u) = \ln(1-u)/l_i(\theta_i), i = 1, \ldots, n$, each of which is the dual stochastic distortion of the stochastic distortion defined in (2)-(5). The reason for selecting the dual form follows from Table 1, where $T^{(i)}_{\widehat{\omega}_i(u)}$ tends to $T^{(i)}_0$ when u tends to zero. Assume that $b = 0$ in (5). Using Proposition 1.9 in Page 285 of Revuz and Yor (1999), the infinitesimal generator of the Lévy subordinator $T^{(i)}_t$ satisfies that

$$\mathscr{A}_{T^{(i)}_\cdot} f(x) = \lim_{t\to 0^+} \frac{1}{t}\left(E\left[f\left(T^{(i)}_t + x\right)\right] - f(x)\right) = \int_0^\infty \left(f(x+y) - f(x)\right)\nu_i(dy), x > 0 \tag{16}$$

for $f \in \mathscr{C}_0$, where ν_i is the Lévy measure of the subordinator $T^{(i)}_t$. Further the infinitesimal generator of $T^{(i)}_{\widehat{\omega}_i(u)}$ is

$$\begin{aligned}
\mathscr{A}_{T^{(i)}_{\widehat{\omega}_i(\cdot)}} f(x) &= \lim_{u\to 0^+} \frac{1}{u}\left(E\left[f\left(T^{(i)}_{\widehat{\omega}_i(u)} + x\right)\right] - f(x)\right) \\
&= \mathscr{A}_{T^{(i)}_\cdot} f(x) \cdot \widehat{\omega}_i'(0) = -\frac{\int_0^\infty \left(f(x+y) - f(x)\right)\nu_i(dy)}{l_i(\theta_i)}, x > 0.
\end{aligned} \tag{17}$$

When the Lévy subordinator distortions $X_i, i = 1, \ldots, n$ are independent, we can calculate the density function of the stochastic distorted copula according the following procedure. For the base copula B and the independent Lévy subordinator distortions $X_i, i = 1, \ldots, n$, we denote

$$h_1(x_1; X_2(u_2), \ldots, X_n(u_n)) = B(1, X_2(u_2), \ldots, X_n(u_n)) - B(1 - \exp(\theta_1 x_1), X_2(u_2), \ldots, X_n(u_n)), x_1 \geq 0.$$

For simplicity, we use $h_1(x_1)$ to represent $h_1(x_1; X_2(u_2), \ldots, X_n(u_n))$ in the following. Since $h_1 \in \mathscr{C}_0$, we know that $h_1 \in \mathscr{D}\left(\mathscr{A}_{T^{(1)}_\cdot}\right)$. Then by the Lipschitz continuity of copula function, we have

$$\begin{aligned}
& \frac{1}{|s|}\left|E\left[h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1+s)}\right) - h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1)}\right)\Big| X_2(u_2), \ldots, X_n(u_n)\right]\right| \\
\leq\ & \frac{1}{|s|}E\left[\left|h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1+s)}\right) - h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1)}\right)\right|\Big| X_2(u_2), \ldots, X_n(u_n)\right] \\
=\ & \frac{1}{|s|}E\left[\left|B\left(1 - \exp\left(\theta_1 T^{(1)}_{\widehat{\omega}_1(u_1+s)}\right), X_2(u_2), \ldots, X_n(u_n)\right)\right.\right. \\
& \left.\left. - B\left(1 - \exp\left(\theta_1 T^{(1)}_{\widehat{\omega}_1(u_1)}\right), X_2(u_2), \ldots, X_n(u_n)\right)\right|\Big| X_2(u_2), \ldots, X_n(u_n)\right] \\
\leq\ & \frac{1}{|s|}E\left[\left|\exp\left(\theta_1 T^{(1)}_{\widehat{\omega}_1(u_1)}\right) - \exp\left(\theta_1 T^{(1)}_{\widehat{\omega}_1(u_1+s)}\right)\right|\right].
\end{aligned}$$

Applying $ET^{(1)}_{\widehat{\omega}_1(u_1)} = u_1, ET^{(1)}_{\widehat{\omega}_1(u_1+s)} = u_1 + s$, we have

$$E\left[|\exp\left(\theta_1 T^{(1)}_{\widehat{\omega}_1(u_1)}\right) - \exp\left(\theta_1 T^{(1)}_{\widehat{\omega}_1(u_1+s)}\right)|\right] \leq E\left[|T^{(1)}_{\widehat{\omega}_1(u_1)} - T^{(1)}_{\widehat{\omega}_1(u_1+s)}|\right] = |s|.$$

Thus combining the above inequalities, we show that

$$\frac{1}{|s|}\left|E\left[h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1+s)}\right) - h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1)}\right)|X_2(u_2),\ldots,X_n(u_n)\right]\right| \leq 1.$$

Using the dominated convergence theorem, the first-order partial derivative of the stochastic distorted copula satisfies that

$$\begin{aligned}
\frac{\partial}{\partial u_1}C^{\mathbf{X}|B}(u_1,\ldots,u_n) &= -\lim_{s\to 0^+}\frac{1}{s}E\left[h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1+s)}\right) - h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1)}\right)\right] \\
&= -\lim_{s\to 0^+}\frac{1}{s}E\left[E\left[h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1+s)}\right) - h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1)}\right) \mid X_2(u_2),\ldots,X_n(u_n)\right]\right] \\
&= -E\left[\lim_{s\to 0^+}\frac{1}{s}E\left[h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1+s)}\right) - h_1\left(T^{(1)}_{\widehat{\omega}_1(u_1)}\right) \mid X_2(u_2),\ldots,X_n(u_n)\right]\right].
\end{aligned}$$

Given $X_2(u_2),\ldots,X_n(u_n)$, the function h_1 belongs to the domain of the infinitesimal generator of $T_{\widehat{\omega}_1(u_1)}$, thus by Proposition 1.2 in Page 282 of Revuz and Yor (1999) we have

$$\begin{aligned}
&\frac{\partial}{\partial u_1}C^{\mathbf{X}|B}(u_1,\ldots,u_n) \\
=\ & E\left[\int_0^\infty (B(1-e^{\theta_1(T^{(1)}_{\widehat{\omega}_1(u_1)}+y)}, X_2(u_2),\ldots,X_n(u_n)) - B(1-e^{\theta_1 T^{(1)}_{\widehat{\omega}_1(u_1)}}, X_2(u_2),\ldots,X_n(u_n)))\nu_1(dy)\right] \\
&\cdot\frac{-1}{l_1(\theta_1)},
\end{aligned}$$

where $l_1(\theta)$ is the Laplace exponent of $T^{(1)}_t$. Repeating the above procedure, we can obtain the density function of the Lévy subordinator distorted copula.

The density expression of the bivariate case is given in the following proposition by using the above idea. A detailed proof will be given in the appendix. In the following, we denote by $V_B([\mathbf{a},\mathbf{b}])$ the probability volume of copula B on the rectangular $[\mathbf{a},\mathbf{b}]$.

Proposition 3.2. *Consider the bivariate Lévy subordinator distorted copula*

$$C^{\mathbf{X}|B}(u_1,u_2) = E\left[B\left(X_1(u_1), X_2(u_2)\right)\right], u_1,u_2 \in [0,1],$$

where $X_i, i = 1,2$ are the dual version of Lévy subordinator distortions defined in (2)-(5) with $b = 0$. If the stochastic distortions X_1 and X_2 are independent, then the density function $c^{\mathbf{X}|B}$ of $C^{\mathbf{X}|B}$ can be expressed as

$$c^{\mathbf{X}|B}(u_1,u_2) = \frac{1}{l_1(\theta_1)l_2(\theta_2)}E\left[\int_0^\infty\int_0^\infty V_B\left([\mathbf{a}(u_1,u_2),\mathbf{b}(u_1,u_2;y_1,y_2)]\right)\nu_1(dy_1)\nu_2(dy_2)\right], u_1,u_2 \in (0,1),$$

where

$$\mathbf{a}(u_1,u_2) = \left(1-e^{\theta_1 T^{(1)}_{\widehat{\omega}_1(u_1)}}, 1-e^{\theta_2 T^{(2)}_{\widehat{\omega}_2(u_2)}}\right), \mathbf{b}(u_1,u_2;y_1,y_2) = \left(1-e^{\theta_1(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1)}, 1-e^{\theta_2(T^{(2)}_{\widehat{\omega}_2(u_2)}+y_2)}\right).$$

The above proposition states that if the stochastic distortions are mutually independent, the family of Lévy subordinator distortions defined in (2)-(5) with $b=0$ can guarantee the existence of the density function even if the base copula is not differentiable.

Generally, given the density function $c^{\mathbf{X}|B}$ of the stochastic distorted copula $C^{\mathbf{X}|B}$, for any $\mathfrak{A}\in\mathscr{B}\left([0,1]^n\right)$ we have

$$\int_{\mathfrak{A}} c^{\mathbf{X}|B}(\mathbf{u})\,d\mathbf{u} = P\left(\left(X_1^{-1}(U_1),\ldots,X_n^{-1}(U_n)\right)\in\mathfrak{A}\right) = P\left((U_1,\ldots,U_n)\in\mathbf{X}(\mathfrak{A})\right),$$

where $(U_1,\ldots,U_n)$ has distribution B, $\mathbf{X}(\mathfrak{A})=\{(X_1(u_1),\ldots,X_n(u_n)):\,(u_1,\ldots,u_n)\in\mathfrak{A}\}$ and $(U_1,\ldots,U_n)$ is independent of $(X_1(u_1),\ldots,X_n(u_n))$.

3.3 Tail dependence of stochastic distorted copula

Consider the stochastic distorted copula with the base copula B and the multi-dimensional stochastic distortion $\mathbf{X}(u)=(X_1(u),X_2(u),\ldots,X_n(u))$. The tail dependence coefficient of the stochastic distorted copula can be directly expressed by the infinitesimal generator of $\mathbf{X}(u)=(X_1(u),X_2(u),\ldots,X_n(u))$.

When the multi-dimensional stochastic distortion $\mathbf{X}(u)=(X_1(u),X_2(u),\ldots,X_n(u))$ is a Feller process with $E[X_i(u)]=u$, $i=1,\ldots,n$, then the lower tail dependence coefficient of the stochastic distorted copula $C^{\mathbf{X}|B}(u_1,\ldots,u_n)$ is given by

$$\lambda^L_{C^{\mathbf{X}|B}} \triangleq \lim_{u\to0^+}\frac{C^{\mathbf{X}|B}(u,\ldots,u)}{u} = \lim_{u\to0^+}\frac{E\left[B\left(X_1(u),X_2(u),\ldots,X_n(u)\right)\right]}{u} \tag{18}$$

whenever the limit exists.

For a stochastic distortion constructed from the Lévy subordinator, the lower tail dependence coefficient has the following expression.

Proposition 3.3. *Consider the Lévy subordinator distorted copula*

$$C^{\mathbf{X}|B}(u_1,\ldots,u_n)=E\left[B\left(X_1(u_1),\ldots,X_n(u_n)\right)\right],u_1,\ldots,u_n\in[0,1]$$

where $\theta<0$,

$$X_i(u)=1-\exp\left(\theta S^{(i)}_{\widehat{\omega}(u)}\right),u\in(0,1),\quad \textit{and}\quad X_i(0)=\lim_{u\to0^+}X_i(u),\quad X_i(1)=\lim_{u\to1^-}X_i(u),$$

here $\widehat{\omega}(u)=\ln(1-u)/l(\theta)$ and $l(\theta)=\gamma\theta+\int_0^\infty\left(e^{\theta x}-1-x1_{|x|\le1}\right)\nu(dx)$ is the common Laplace exponent of the Lévy subordinator $S^{(i)}_t, i=1,\ldots,n$. When the base copula B is differentiable at $\mathbf{0}=(0,\ldots,0)$, the lower tail dependence coefficient of $C^{\mathbf{X}|B}$ can be expressed as

$$\lambda^L_{C^{\mathbf{X}|B}}=\frac{1}{l(\theta)}\left(\gamma\theta\sum_{i=1}^n\frac{\partial B}{\partial u_i}(\mathbf{0})-\int_{\mathbb{R}^n_+}\left(B\left(1-e^{\theta y_1},\ldots,1-e^{\theta y_n}\right)+\sum_{i=1}^n\theta\frac{\partial B}{\partial u_i}(\mathbf{0})\,y_i1_{||\mathbf{y}||\le1}\right)\mathbf{N}(d\mathbf{y})\right), \tag{19}$$

where $\mathbf{N}$ is the joint Lévy measure of $\mathbf{S}_t=\left(S^{(1)}_t,\ldots,S^{(n)}_t\right)$.

Proof. From (18), we can express the lower tail dependence coefficient of the Lévy subordinator distorted copula as

$$\lambda^L_{C^{\mathbf{X}|B}}=-\lim_{t\to0^+}\frac{E\left[H\left(\mathbf{S}_{\widehat{\omega}(t)}\right)-H(\mathbf{0})\right]}{t}=-\lim_{t\to0^+}\frac{E\left[H(\mathbf{S}_t)-H(\mathbf{0})\right]}{t}\widehat{\omega}'(0)=-\mathscr{A}H(\mathbf{0})\,\widehat{\omega}'(0), \tag{20}$$

where

$$H(x_1,\ldots,x_n) = 1 - B(1-\exp(\theta x_1),\ldots,1-\exp(\theta x_n)), \quad \lim_{||\mathbf{x}||\to\infty} H(\mathbf{x}) = 0, \tag{21}$$

and $\mathscr{A}$ is the infinitesimal generator of $\mathbf{X}(u)$ defined in (15). Since the base copula B is continuous and $\lim_{||\mathbf{x}||\to\infty} H(\mathbf{x}) = 0$, we know H belongs to the domain of the infinitesimal generator of $\mathbf{X}(u)$. The purpose that we introduce H here is to make a transformation for B so that the function H belongs to the domain of the infinitesimal generator of $\mathbf{X}(u)$.

Using Proposition 3.16 in Cont and Tankov (2004) we know that the infinitesimal generator of $\mathbf{S}_t = \left(S_t^{(1)},\ldots,S_t^{(n)}\right)$ equals to

$$\begin{aligned}
\mathscr{A}H(\mathbf{x}) &= \lim_{t\to 0^+} \frac{1}{t}\left(E\left[H(\mathbf{S}_t+\mathbf{x})\right] - H(\mathbf{x})\right) \\
&= \gamma \sum_{i=1}^n \frac{\partial H}{\partial x_i}(\mathbf{x}) + \int_{\mathbb{R}_+^n} \left(H(\mathbf{x}+\mathbf{y}) - H(\mathbf{x}) - \sum_{i=1}^n \frac{\partial H}{\partial x_i}(\mathbf{x})\, y_i 1_{||\mathbf{y}||\le 1}\right) \mathbf{N}(d\mathbf{y}).
\end{aligned} \tag{22}$$

Combining (20)-(22) and $\widehat{\omega}'(0) = -\frac{1}{l(\theta)}$, we get (19). □

Since $\int_{\mathbb{R}_+^n} (||\mathbf{x}|| \wedge 1)\,\mathbf{N}(d\mathbf{x}) < \infty$ and $\frac{\partial B}{\partial u_i}(\mathbf{0}), i = 1,\ldots,n$ are bounded, the integral (19) is finite. Moreover, when the subordinators are pure jump processes (i.e., zero drift speeds) and they are mutually independent, the lower tail dependence of the subordinator distorted copula vanishes even if the base copula has tail dependence.

4 Some families of stochastic distorted copulas

As noted before, the stochastic distorted copula defined in (8) consists of two components: the base copula function B and the stochastic distortions $X_i, i \le n$. In this section, we will give some families of stochastic distorted copulas to demonstrate the versatility of stochastic distortions.

It is argued that FGM copula, comonotonic copula and countermonotonic copula are three important copulas in modeling dependence. For example, FGM copula has been widely-used due to its simple polynomial form (Nelsen, 2006); Comonotonicity and countermonotonicity are important dependence structures used in finance and insurance (Deelstra et al., 2011; Yang et al., 2009). In the following, first we choose FGM copula as the base copula to illustrate the influence of the stochastic distortions on correlation. Second we use the comonotonic copula and countermonotonic copula as the base copula to show the effect of stochastic distortions on copula's desingularization. Finally, by choosing the product copula as the base copula and applying the factor-structured stochastic distortions we present one type of stochastic distorted copulas to reflect the correlation structure of a linear factor model.

In the following, we focus our discussion on the stochastic distortions constructed from Lévy subordinators.

4.1 FGM copula as the base copula

The n-dimensional FGM copula is given by

$$C_{FGM}(u_1,\ldots,u_n) = \prod_{i=1}^n u_i \left(1 + \sum_{k=2}^n \sum_{1\le j_1<\cdots<j_k\le n} \eta_{j_1 j_2\cdots j_k} \bar{u}_{j_1}\bar{u}_{j_2}\cdots\bar{u}_{j_k}\right), u_1,\ldots,u_n \in [0,1], \tag{23}$$

where $\bar{u}_i = 1 - u_i, i \leq n$ and $|\eta_{j_1 j_2 \cdots j_k}| \leq 1$ for all $1 \leq j_1 < \cdots < j_k \leq n$. When FGM copula is chosen as the base copula and the marginal stochastic distortions are mutually independent or the same, the corresponding stochastic distorted copula is given in the following proposition. The proof will be given in the appendix.

Proposition 4.1. *Assume that the base copula is chosen as FGM copula in (23), and the multivariate stochastic distortions are Lévy subordinator distortions*

$$X_i(u) = \exp\left(\theta_i T^{(i)}_{\omega_i(u)}\right), u \in (0,1), \quad \textit{and} \quad X_i(0) = \lim_{u \to 0^+} X_i(u), \quad X_i(1) = \lim_{u \to 1^-} X_i(u),$$

where for each $i = 1, \ldots, n$, the parameter $\theta_i < 0$, $\omega_i(u) = \ln(u)/l_i(\theta_i)$ and $T^{(i)}_t$ is the Lévy subordinator with Laplace exponent $l_i(\theta_i)$.
(a) If the Lévy subordinators $T^{(i)}_t, i = 1, \ldots, n$ are independent, then the stochastic distorted copula is given by

$$\begin{aligned} & C^{\Pi}_{FGM}(u_1, \ldots, u_n) \\ & = \prod_{i=1}^{n} u_i \left[1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \cdots < j_k \leq n} \eta_{j_1 j_2 \cdots j_k} \prod_{m=1}^{k} \left(1 - u_{j_m}^{l_m(2\theta_{j_m})/l_m(\theta_{j_m}) - 1}\right)\right], u_1, \ldots, u_n \in [0,1]. \end{aligned} \quad (24)$$

(b) In the two-dimensional case, if $\theta_1 = \theta_2 = \theta < 0$ and the Lévy subordinators $T^{(1)}_t = T^{(2)}_t$ with the same Laplace exponent $l(\theta)$, then the two-dimensional stochastic distorted copula is given by

$$\begin{aligned} & C^{M}_{FGM}(u_1, u_2) = (\eta_{12} + 1) u_{(1)}^{\frac{l(2\theta)}{l(\theta)} - 1} u_{(2)} \\ & + \eta_{12} \left[u_{(1)}^{\frac{l(4\theta) - l(2\theta)}{l(\theta)}} u_{(2)}^{\frac{l(2\theta)}{l(\theta)}} - u_{(1)}^{\frac{l(3\theta) - l(\theta)}{l(\theta)}} u_{(2)} - u_{(1)}^{\frac{l(3\theta) - l(2\theta)}{l(\theta)}} u_{(2)}^{\frac{l(2\theta)}{l(\theta)}}\right], u_1, u_2 \in [0,1], \end{aligned} \quad (25)$$

where $u_{(1)} = u_1 \vee u_2$, $u_{(2)} = u_1 \wedge u_2$.

Proposition 4.1 shows that when the marginal stochastic distortions are independent, the corresponding stochastic distorted copula has a simple expression. In the bivariate case, the stochastic distorted copula in (24) can be expressed as

$$C^{\Pi}_{FGM}(u_1, u_2) = u_1 u_2 \left[1 + \eta_{12}\left(1 - u_1^{l_1(2\theta_1)/l_1(\theta_1) - 1}\right)\left(1 - u_2^{l_2(2\theta_2)/l_2(\theta_2) - 1}\right)\right], \quad (26)$$

and its Spearman's ρ equals to

$$\rho^{\Pi}_{FGM} = \eta_{12}\left[3 - 6\frac{l_1(2\theta_1)/l_1(\theta_1) + l_2(2\theta_2)/l_2(\theta_2)}{(1 + l_1(2\theta_1)/l_1(\theta_1))(1 + l_2(2\theta_2)/l_2(\theta_2))}\right].$$

Since $l_i(\theta)$ is nondecreasing and convex with $l_i(0) = 0$, we have $1 \leq l_i(2\theta)/l_i(\theta) \leq 2$. Further it can be verified that

$$0 \leq 3 - 6\frac{l_1(2\theta_1)/l_1(\theta_1) + l_2(2\theta_2)/l_2(\theta_2)}{(1 + l_1(2\theta_1)/l_1(\theta_1))(1 + l_2(2\theta_2)/l_2(\theta_2))} \leq \frac{1}{3}.$$

The first equality holds when $l_1(2\theta_1)/l_1(\theta_1) = l_2(2\theta_2)/l_2(\theta_2) = 1$, and the second equality holds when $l_1(2\theta_1)/l_1(\theta_1) = l_2(2\theta_2)/l_2(\theta_2) = 2$. Thus if $\eta_{12} \geq 0$, ρ^{Π}_{FGM} in the bivariate case satisfies that

$$0 \leq \rho^{\Pi}_{FGM} \leq \frac{1}{3}\eta_{12} = \rho_{FGM}$$

and if $\eta_{12} < 0$,

$$\rho_{FGM} = \frac{1}{3}\eta_{12} \leq \rho_{FGM}^{\Pi} \leq 0,$$

where ρ_{FGM} denotes the Spearman's ρ of the bivariate FGM copula. The above inequalities show that when the base copula is chosen as FGM copula, the independent stochastic distortions have the effect of reducing the correlation of the base copula. In addition, it is easy to verify that the stochastic distorted copula (26) has no upper and lower tail dependence.

For the tempered stable subordinator (7), we have that for $m > 0$,

$$l_i(m\theta_i)/l_i(\theta_i) = \begin{cases} \frac{(1-\alpha_i)^{\alpha_i} - (1-\alpha_i - m\kappa_i\theta_i)^{\alpha_i}}{(1-\alpha_i)^{\alpha_i} - (1-\alpha_i - \kappa_i\theta_i)^{\alpha_i}}, & 0 < \alpha_i < 1, \\ \frac{\ln(1-m\kappa_i\theta_i)}{\ln(1-\kappa_i\theta_i)}, & \alpha_i = 0, \end{cases} \tag{27}$$

and

$$\lim_{\kappa_i \to 0} l_i(m\theta_i)/l_i(\theta_i) = m, \ \lim_{\kappa_i \to \infty} l_i(m\theta_i)/l_i(\theta_i) = m^{\alpha_i}, \tag{28}$$

where κ_i and α_i are the parameters in $l_i(\theta_i)$. In this case, the stochastic distorted FGM copula (26) has the following limits

$$\lim_{\kappa_1,\kappa_2 \to 0} C_{FGM}^{\Pi}(u_1,u_2) = C_{FGM}(u_1,u_2), 0 \leq \alpha_i < 1,$$

$$\lim_{\kappa_1,\kappa_2 \to \infty} C_{FGM}^{\Pi}(u_1,u_2) = \Pi(u_1,u_2), \alpha_1 \cdot \alpha_2 = 0.$$

Therefore, as the variance parameters κ_1 and κ_2 of the time-change processes tend to zero, that is, the stochastic effect vanishes, the stochastic distorted copula tends to the base copula. When both κ_1 and κ_2 go to infinity, the stochastic distorted copula converges to the independent copula for α_1, α_2 satisfying $\alpha_1 \cdot \alpha_2 = 0$.

Also applying (27) and (28) to the stochastic distorted copula (25), we have

$$\lim_{\kappa \to 0} C_{FGM}^{M}(u_1,u_2) = C_{FGM}(u_1,u_2), 0 \leq \alpha < 1,$$

$$\lim_{\kappa \to \infty} C_{FGM}^{M}(u_1,u_2) = \min(u_1,u_2), \alpha = 0,$$

where κ and α are the parameters in $l(\theta)$. The stochastic distorted copula tends to the base copula when κ goes to zero, and the stochastic distorted copula tends to the comonotonic copula for $\alpha = 0$ when κ goes to infinity. Thus the same stochastic distortions can distort the negative correlation copula to the comonotonic copula, which reflects the strong capability of the stochastic distortions.

4.2 Comonotonic copula and countermonotonic copula as the base copulas

As mentioned above, comonotonic copula and countermonotonic copula are two important families in modeling dependence in finance and insurance. In this subsection, we will apply independent inverse Gaussian processes to comonotonic copula and countermonotonic copula to obtain the analytic expression of the corresponding stochastic distorted copulas, which shows the desingularization effect of stochastic distortions.

Consider the stochastic distortions defined in (2)-(5). For each fixed $i = 1, \ldots, n$ and $\theta_i < 0$, let

$$X_i(u) = \exp\left(\theta_i T_{\omega_i(u)}^{(i)}\right), u \in (0,1), \quad \text{and} \quad X_i(0) = \lim_{u \to 0^+} X_i(u), \quad X_i(1) = \lim_{u \to 1^-} X_i(u), \tag{29}$$

where $\omega_i(u) = \ln(u)/l_i(\theta_i)$ and $T_t^{(i)}$ is an inverse Gaussian process with $E\left[T_t^{(i)}\right] = t, Var\left(T_t^{(i)}\right) = \kappa_i t$ and the Laplace exponent $l_i(\cdot)$. Then $l_i(\theta_i) = \left(1-\sqrt{1-2\kappa_i\theta_i}\right)/\kappa_i$. In addition, each $T_t^{(i)}$ has the explicit density function $p_t^{(i)}(x)$ and distribution function $F_t^{(i)}(x)$ with κ_i replacing κ in the following expressions,

$$p_t(x) = \sqrt{\frac{t^2}{2\pi\kappa x^3}} \exp\left(-\frac{(x-t)^2}{2\kappa x}\right) 1_{x>0}$$

and

$$F_t(x) = \Phi\left(\sqrt{\frac{x}{\kappa}} - \sqrt{\frac{1}{\kappa x}}t\right) + e^{2t/\kappa}\Phi\left(-\sqrt{\frac{x}{\kappa}} - \sqrt{\frac{1}{\kappa x}}t\right), x > 0,$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. When $T_t^{(i)}, i = 1,\ldots,n$ are independent, the corresponding stochastic distorted copula can be expressed as

$$\begin{aligned} C_B^{IG}(u_1,\ldots,u_n) &\triangleq E\left[B\left(\exp\left(\theta_1 T^{(1)}_{\omega_1(u_1)}\right),\ldots,\exp\left(\theta_n T^{(n)}_{\omega_n(u_n)}\right)\right)\right] \\ &= \int_0^\infty \cdots \int_0^\infty B\left(\exp(\theta_1 x_1),\ldots,\exp(\theta_n x_n)\right)\left(\prod_{i=1}^n p^{(i)}_{\omega_i(u_i)}(x_i)\right) dx_1\cdots dx_n. \end{aligned} \tag{30}$$

When the base copula in (30) is chosen as comonotonic copula or countermonotonic copula, the corresponding stochastic distorted copulas are denoted by

$$C_+^{IG}(u_1,\ldots,u_n) \triangleq E\left[C_+\left(\exp\left(\theta_1 T^{(1)}_{\omega_1(u_1)}\right),\ldots,\exp\left(\theta_n T^{(n)}_{\omega_n(u_n)}\right)\right)\right] \tag{31}$$

and

$$C_-^{IG}(u_1,u_2) \triangleq E\left[C_-\left(\exp\left(\theta_1 T^{(1)}_{\omega_1(u_1)}\right),\exp\left(\theta_2 T^{(2)}_{\omega_2(u_2)}\right)\right)\right], \tag{32}$$

respectively. Their analytic formulas are given in the following proposition, and its proof is omitted.

Proposition 4.2. *For the independent stochastic distortions $X_i, i = 1,\ldots,n$ defined in (29) and $u_1,\ldots,u_n \in [0,1]$, we have*

$$C_+^{IG}(u_1,\ldots,u_n) = \int_0^1 \prod_{i=1}^n \left[\Phi\left(a_i\sqrt{-\ln y} - b_i\frac{\ln u_i}{\sqrt{-\ln y}}\right) + \Phi\left(-a_i\sqrt{-\ln y} - b_i\frac{\ln u_i}{\sqrt{-\ln y}}\right)u_i^{2a_ib_i}\right] dy$$

and

$$\begin{aligned} &C_-^{IG}(u_1,u_2) \\ &= u_1 - \int_0^1 \left(\Phi\left(a_1\sqrt{-\ln y} - b_1\frac{\ln u_1}{\sqrt{-\ln y}}\right) + u_1^{2a_1b_1}\Phi\left(-a_1\sqrt{-\ln y} - b_1\frac{\ln u_1}{\sqrt{-\ln y}}\right)\right) \\ &\cdot\left(\Phi\left(-a_2\sqrt{-\ln(1-y)} + b_2\frac{\ln u_2}{\sqrt{-\ln(1-y)}}\right) - u_2^{2a_2b_2}\Phi\left(-a_2\sqrt{-\ln(1-y)} - b_2\frac{\ln u_2}{\sqrt{-\ln(1-y)}}\right)\right) dy, \end{aligned}$$

where $a_i = 1/\sqrt{-\theta_i\kappa_i}, b_i = \sqrt{-\theta_i/\kappa_i}/l_i(\theta_i)$, $i = 1,\ldots,n$.

Proposition 4.2 gives an example for desingularizing the Frechét copula (singular copula) via independent stochastic distortions, which provides enough flexibility to model different kinds of dependence.

Note that a Gamma process also has an explicit density function. If $T_t^{(i)}, i = 1, \ldots, n$ in (29) are independent Gamma process with $E\left[T_t^{(i)}\right] = t$ and $Var\left(T_t^{(i)}\right) = \kappa_i t$, we can similarly get the stochastic distorted copula. Note that if $T_t^{(i)}$ follows the Gamma distribution, its density function $p_t^{(i)}$ and distribution function $F_t^{(i)}$ can be expressed as

$$p_t^{(i)}(x) = \frac{1}{\Gamma(t/\kappa_i)\, k^{t/\kappa_i}} x^{t/\kappa_i - 1} e^{-x/\kappa_i} 1_{x>0}, \quad F_t^{(i)}(x) = \frac{\gamma(t/\kappa_i, x/\kappa_i)}{\Gamma(t/\kappa_i)}, x > 0, \tag{33}$$

where the incomplete Gamma function $\gamma(a,b) = \int_0^b e^{-t} t^{a-1} dt$.

4.3 Stochastic distorted copula generated by linear factor models

Multidimensional subordinators can be used to introduce many features in financial modeling, such as dependent jumps, stochastic volatility, default clustering, etc. As the most convenient class of multidimensional subordinators for practical applications, linear factor models were discussed in Mendoza and Linetsky (2016) for introducing dependency among several stochastic processes. In the following, we will present one family of stochastic distorted copula by applying linear factor models.

For independent Lévy subordinators $S_t^{(j)}, j = 0, 1, \ldots, M$, their Laplace exponents are denoted as $l_j(\theta_j)$, $j = 0, 1, \ldots, M$, respectively. We assume that for $a_{i,j} \geq 0, i = 1 \ldots, n, j = 0, 1, \ldots, M$, $\sum_{m=0}^{M} a_{i,m} = 1$ and

$$T_t^{(i)} = \sum_{j=0}^{M} a_{i,j} S_t^{(j)}, i = 1 \ldots, n. \tag{34}$$

The above linear factor model adopts a parsimonious approach to introduce dependency. See Mendoza and Linetsky (2016) and Barndorff-Nielsen et al. (2001) for details.

Using the time-change processes defined in (34), we can define multivariate stochastic distortions as follows. For fixed $\theta_i < 0, i = 1, \ldots, n$, let $\omega_i(u) = \frac{\ln u}{\sum_{j=0}^{M} l_j(a_{i,j}\theta_i)}, u \in (0,1)$ and

$$X_i(u) = \exp\left(\theta_i T_{\omega_i(u)}^{(i)}\right), u \in (0,1), \quad and \quad X_i(0) = \lim_{u \to 0^+} X_i(u), \quad X_i(1) = \lim_{u \to 1^-} X_i(u). \tag{35}$$

Then $X_i(u)$ is a stochastic distortion satisfying that $E[X_i(u)] = u, u \in [0,1]$. By choosing the product copula $\Pi(u_1, \ldots, u_n) = \prod_{j=1}^{n} u_j$ as the base copula, we have the following stochastic distorted copula. The proof will be given in the appendix.

Proposition 4.3. *When the product copula is chosen as the base copula and the stochastic distortions $X_i(u), i = 1, \ldots, n$ are modeled by applying the linear factor models in (35), the stochastic distorted copula is given by*

$$\Pi_M^{sd}(u_1, \ldots, u_n) = \exp\left(\sum_{j=0}^{M} \sum_{k=1}^{n} \left(\omega_{\pi(k)}\left(u_{\pi(k)}\right) - \omega_{\pi(k-1)}\left(u_{\pi(k-1)}\right)\right) l_j\left(\sum_{i=k}^{n} \theta_{\pi(i)} a_{\pi(i),j}\right)\right), \tag{36}$$

where $u_1, \ldots, u_n \in [0,1]$ and $\left(u_{\pi(1)}, \ldots, u_{\pi(n)}\right)$ is a permutation of $(u_1, \ldots, u_n)$ such that

$$\omega_{\pi(1)}\left(u_{\pi(1)}\right) \leq \omega_{\pi(2)}\left(u_{\pi(2)}\right) \leq \ldots \leq \omega_{\pi(n)}\left(u_{\pi(n)}\right)$$

and $\omega_{\pi(0)}\left(u_{\pi(0)}\right) = 0$.

In this paper, we call the above stochastic distorted copula as the linear factor stochastic distorted copula (LFSDC).

Consider one simplified version of LFSDC. Assume that $S_t^{(j)}, j \geq 1$ have identical Laplace transform, i.e., $l_j(\theta) = l_1(\theta)$ for $j \geq 1$. Let $M = n$, $\theta_i = \theta, a_{i,0} = a, a_{i,i} = b, a_{i,j} = 0$ for $j \neq i$. Then the model (34) can be expressed as

$$T_t^{(i)} = aS_t^{(0)} + bS_t^{(i)},\ a \geq 0, b \geq 0, a+b = 1. \tag{37}$$

In the above model, $S_t^{(0)}$ is the systematic component of $T_t^{(i)}$ and $S_t^{(i)}$ is the idiosyncratic component of $T_t^{(i)}$. Then we have $\omega_i(u) \equiv \omega(u) = \frac{\ln u}{l_0(a\theta)+l_1(b\theta)}$. Thus the stochastic distorted copula (36) can be expressed as

$$\Pi_M^{sd}(u_1,\ldots,u_n) = \prod_{k=1}^{n} \left(\frac{u_{(k)}}{u_{(k-1)}}\right)^{\frac{l_0((n-k+1)\theta a)+(n-k+1)l_1(\theta b)}{l_0(a\theta)+l_1(b\theta)}}, \quad u_1,\ldots,u_n \in [0,1], \tag{38}$$

where $(u_{(1)},\ldots,u_{(n)})$ is a permutation of $(u_1,\ldots,u_n)$ such that $u_{(1)} \geq u_{(2)} \geq \cdots \geq u_{(n)}$ and $u_{(0)} = 1$. When $b = 0$, we have $a = 1$ and the copula function can be simplified as

$$\Pi_M^{sd}(u_1,\ldots,u_n) = \prod_{k=1}^{n} \left(\frac{u_{(k)}}{u_{(k-1)}}\right)^{l_0(\theta(n+1-k))/l_0(\theta)}. \tag{39}$$

Moreover, if the Lévy subordinator $S_t^{(0)}$ is a tempered stable subordinator discussed in Example 2.2 with parameters κ_0 and α_0, by applying (27) and (28) the distorted product copula (39) has the following limits

$$\lim_{\kappa_0 \to 0} \Pi_M^{sd}(u_1,\ldots,u_n) = \Pi(u_1,\ldots,u_n),\ 0 \leq \alpha_0 < 1, \tag{40}$$

$$\lim_{\kappa_0 \to \infty} \Pi_M^{sd}(u_1,\ldots,u_n) = C_+(u_1,\ldots,u_n),\ \alpha_0 = 0. \tag{41}$$

The limit (40) states that when the variance of the time-change process tends to zero the stochastic distorted copula converges to the product copula. The limit (41) states that when the time-change process is a Gamma process, i.e., $\alpha_0 = 0$, the stochastic distorted copula (39) converges to the comonotonic copula as the variance of the Gamma process tends to infinity. Note that the product copula $\Pi(u_1,\ldots,u_n)$ has zero tail dependence and the comonotonic copula $C_+(u_1,\ldots,u_n)$ has perfect tail dependence. Thus this example shows that the dependent stochastic distortions can change the tail dependence of the base copula. Specially, in the two-dimensional case the upper tail dependence coefficient of the copula Π_M^{sd} in (39) can be expressed as

$$\lambda^U_{\Pi_M^{sd}} = \lim_{u \to 0^+} \frac{\hat{\Pi}_M^{sd}(u,u)}{u} = 2 - \frac{l_0(2\theta)}{l_0(\theta)},$$

where $\hat{\Pi}_M^{sd}$ is the survival copula of Π_M^{sd}.

5 Application in portfolio credit risk

5.1 Correlation structure among default times

In this section, we will apply the linear factor stochastic distorted copula (LFSDC) in Proposition 4.3 to model the default correlation in a credit portfolio. The advantage of using LFSDC is that it has tail dependence, and the structure of the linear factor model is implied by the copula. Note that the idea of using a linear factor model is widely employed in modeling credit risk (Laurent and Gregory, 2005; Burtschell et al., 2012; Hull and White, 2004).

Suppose we observe n correlated default times $\tau_1,\ldots,\tau_n$ in the calendar time for n credit entities with marginal distributions $F_i, i=1,\ldots,n$. In order to model the dependency among the default times, we introduce random factors $T_t^{(1)},\ldots,T_t^{(n)}$. Assume that $T_t^{(1)},\ldots,T_t^{(n)}$ are modeled as combinations of systematic factor and idiosyncratic factors in (37) with $a=\rho, b=1-\rho$, where $\left(S_t^{(0)},S_t^{(1)},\ldots,S_t^{(n)}\right)$ are $n+1$ independent Gamma processes. Here $S_t^{(0)}$ is the systematic factor with $E\left[S_t^{(0)}\right]=t, Var\left[S_1^{(0)}\right]=\kappa_0$ and the Laplace exponent $l_0(\theta)=-\ln(1-\kappa_0\theta)/\kappa_0, \theta\le 0$, and $S_t^{(1)},\ldots,S_t^{(n)}$ are the idiosyncratic factors with common $E\left[S_t^{(j)}\right]=t$, $Var\left[S_1^{(j)}\right]=\kappa_1$ and the Laplace exponent $l_j(\theta)=l_1(\theta)=-\ln(1-\kappa_1\theta)/\kappa_1, \theta\le 0$, $j=1,\ldots,n$. Then $T_t^{(1)},\ldots,T_t^{(n)}$ have the same Laplace exponent $l(\theta)=l_0(\rho\theta)+l_1((1-\rho)\theta)$. For each fixed i, let $\omega(u)=\ln(u)/l(\theta)$ and

$$X_i(u)=\exp\left(\theta T^{(i)}_{\omega(u)}\right), u\in(0,1), \quad and \quad X_i(0)=\lim_{u\to 0^+}X_i(u), X_i(1)=\lim_{u\to 1^-}X_i(u).$$

Note that $\widehat{X}_i(u)=1-X_i(1-u)$ is the dual stochastic distortion of $X_i(u)$.

We introduce i.i.d. uniform [0,1] random variables $U_1,\ldots,U_n$, which are independent of the stochastic distortions $\widehat{X}_1(u),\ldots,\widehat{X}_n(u)$. Then $\widehat{X}_1^{-1}(U_1),\ldots,\widehat{X}_n^{-1}(U_n)$ are uniform [0,1] random variables. Assume that the marginal distributions $F_i, i=1,\ldots,n$ are continuous, and the default times $\tau_1,\ldots,\tau_n$ are defined as

$$\tau_1=F_1^{-1}(\widehat{X}_1^{-1}(U_1)),\ldots,\tau_n=F_n^{-1}(\widehat{X}_n^{-1}(U_n)).$$

Then we have

$$F_1(\tau_1)=\widehat{X}_1^{-1}(U_1),\ldots,F_n(\tau_n)=\widehat{X}_n^{-1}(U_n)$$

and

$$\left(\widehat{X}_1(F_1(\tau_1)),\ldots,\widehat{X}_n(F_n(\tau_n))\right)=(U_1,\ldots,U_n). \tag{42}$$

Let C be the copula function of $(\tau_1,\ldots,\tau_n)$. Then we have

$$C(u_1,\ldots,u_n)=E\left[\prod_{i=1}^n\widehat{X}_i(u_i)\right]=\widehat{\Pi_M^{sd}}(u_1,\ldots,u_n), \tag{43}$$

where $\widehat{\Pi_M^{sd}}$ is the survival copula of Π_M^{sd} in (38). By (38), we know that each bivariate marginal copula of $C(u_1,\ldots,u_n)$ has the lower tail dependence coefficient

$$\lambda_C^L=2-\frac{\ln(1-2\rho\theta\kappa_0)\kappa_1+2\ln(1-2(1-\rho)\kappa_1)\kappa_0}{\ln(1-\rho\theta\kappa_0)\kappa_1+\ln(1-(1-\rho)\kappa_1)\kappa_0}. \tag{44}$$

For simplicity, we call the above copula function as LFSDC with one factor Gamma process.

Equation (42) implies that $T^{(1)}_{\omega(1-F_1(\tau_1))},\ldots,T^{(n)}_{\omega(1-F_n(\tau_n))}$ are independent. In other words, there exist appropriate time-change transformations such that the transformed default times are mutually independent. And we have

$$\omega(1-F_i(\tau_i))=\left(T^{(i)}\right)^{-1}\left(\frac{1}{\theta}\ln(1-U_i)\right), i=1,\ldots,n. \tag{45}$$

We can simulate the above model by the following steps:

(1) Generate n independent identically Uniform [0,1] random variables $U_1,\ldots,U_n$;

(2) Calculate $\xi_1,\ldots,\xi_n$ by $\xi_i = \left(T^{(i)}\right)^{-1}\left(\frac{1}{\theta}\ln(1-U_i)\right), i=1,\ldots,n$. A simple way to generate the first cross time $\left(T^{(i)}\right)^{-1}\left(\frac{1}{\theta}\ln(1-U_i)\right)$ is to select a small enough time step Δ, then simulate Gamma distributed increment $T^{(i)}(n\Delta)-T^{(i)}((n-1)\Delta)$ and accumulate path $T^{(i)}(n\Delta)$ until the accumulated path cross over $\frac{1}{\theta}\ln(1-U_i)$. Hence the first cross time $N_i\Delta$ is an approximation of $\left(T^{(i)}\right)^{-1}\left(\frac{1}{\theta}\ln(1-U_i)\right)$, i.e.,

$$N_i = \inf\left\{K : \sum_{n=1}^{K}\left(T^{(i)}(n\Delta)-T^{(i)}((n-1)\Delta)\right) \geq \frac{1}{\theta}\ln(1-U_i)\right\}.$$

Then let $\xi_i = N_i\Delta$. For a detailed introduction on generating Gamma distributed random variables and Gamma processes, see Chapter 6 in Cont and Tankov (2004).
(3) Use (45) to obtain the default times $\tau_1,\ldots,\tau_n$ by $\tau_i = F_i^{-1}\left(1-\omega^{-1}(\xi_i)\right), i=1,\ldots,n$.

Denote by $N(s) = \sum_{i=1}^{n} 1_{\{\tau_i \leq s\}}$ the number of defaults in the credit portfolio until time s. Given the systematic factor $S_t^{(0)}, t \geq 0$, the default times are conditionally independent and the probability generating function of $N(s)$ can be expressed as

$$\psi_{N(s)}(z) = E\left[z^{N(s)}\right] = E\left[E\left[z^{N(s)} \mid S_{\cdot}^{(0)}\right]\right] = E\left[\prod_{i=1}^{n}\left[z+(1-z)P\left(\tau_i > s \mid S_{\cdot}^{(0)}\right)\right]\right],$$

where the conditional survival function can be expressed as

$$P\left(\tau_i > s \mid S_{\cdot}^{(0)}\right) = \exp\left(\theta\rho S^{(0)}_{\omega(1-F_i(s))}\right)\exp\left(\omega(1-F_i(s))\, l_1((1-\rho)\theta)\right).$$

The probability function of $N(s)$ is important in collateralized debt obligation (CDO) pricing. For details on CDO pricing methodology, we refer to the Appendix.

5.2 Numerical results on probability function of the number of defaults

In the next, we study the distribution of the number of defaults $N(5)$ under our model through a numerical example. Here θ is set to be -1. Consider a credit portfolio with 100 entities and the marginal distributions of default times are assumed to be exponentially distributed with intensity parameters

$$\lambda_i = 0.02 + 0.001 \times (i-1),\ i = 1,\ldots,100.$$

The numerical results of the probability function $P(N(5)=n)$ are presented in Figures 1 and 2.

Note that ρ, κ_0 and κ_1 represent the systematic factor weight, the variance of the systematic Gamma variable $S_1^{(0)}$ and the variance of the idiosyncratic Gamma variable $S_1^{(1)}$, respectively. From Figure 1 we can see that as the systematic factor weight ρ increases, the probability function of the number of defaults moves to the left but it keeps the right tail decreasing. From Figure 2 we can see that as the variances κ_0, κ_1 increase, the probability function of the number of defaults $N(5)$ squeezes to both left side and right side. Note that this property is very favorable for calibrating CDO market data (Hull and White, 2004). Figure 2 also manifests that our model can produce an increasing tail for the probability function of the number of defaults, which reflects the tail dependence and default clustering.

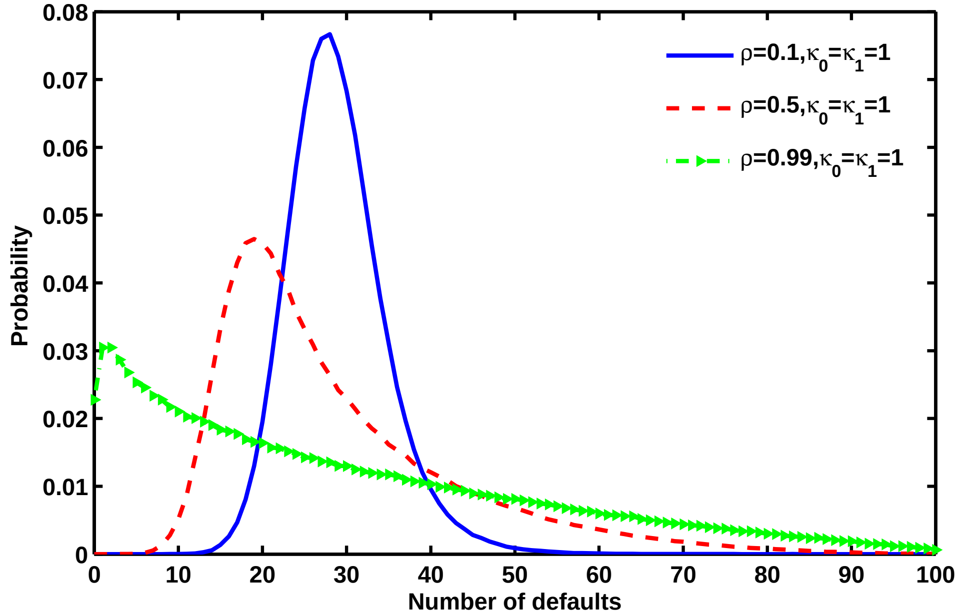

Figure 1: The impact of the systematic factor weight ρ on the probability function of the number of defaults.

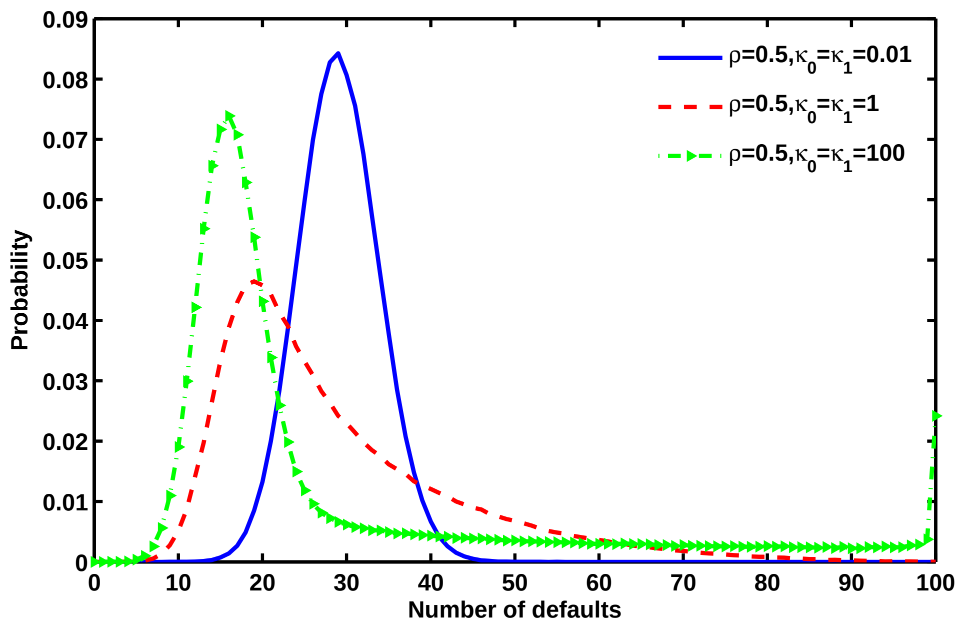

Figure 2: The impact of the variances κ_0, κ_1 of the Gamma processes on the probability function of the number of defaults.

5.3 Market data calibration on CDO

A CDO is a type of structured asset-backed security. It can be thought of as a promise to pay investors in a prescribed sequence. Based on the cash flow, the CDO collects from the pool of bonds or other assets it owns. Default correlation is essential in pricing a CDO tranche. The market benchmark pricing model for CDO tranches is the one factor Gaussian copula model, see Hull and White (2004).

We proceed to apply our LFSDC with one factor gamma process to the synthetic CDO market data to model the correlation between default times. In the following, θ is set to be -1 again. We look at the iTraxx Europe S24 5-year tranches at April 25, 2016, which is based on 125 names. Table 2 gives the basic tranches pricing information. The market data can be obtained from Bloomberg, which is listed in the appendix.

Pricing Date	Reference Portfolio	Maturity
4-25-2016	iTraxx Europe S24 5-year	12-20-2020

Table 2: iTraxx Europe S24 pricing information

It is assumed that the marginal default times are exponentially distributed. We firstly calculate the linear interpolated CDS spread according to the time to maturity of index tranches and then imply the marginal intensity from the interpolated CDS spread for each name. We assume a recovery rate of 40% and adopt Europe standard swap rate curve from Bloomberg to discount cash flows. For a detailed introduction on CDS valuation, see O'Kane and Turnbull (2003).

In order to make comparisons with other models in the literature, we did not optimize the whole parameters to perfectly match the market price. Instead we simply calibrate the systematic factor weight such that the model price matches the market quote of equity tranche with given variance parameters of the Gamma processes. The calibration results are presented in Tables 3-5. In these tables, notation "$\Gamma(\kappa_0)-\Gamma(\kappa_1)$" represents our LFSDC with one factor gamma process (43), where κ_0 and κ_1 are the variance parameters of the systematic Gamma process and the idiosyncratic Gamma process, respectively. The market quotes for the first two tranches are up-front quotes, which are different with the spread quotes of the next two tranches. For instance, the up-front quote of 38.75% for the equity tranche $[0\%,3\%]$ means that the protection seller receives the running spread quarterly on the outstanding principal plus an initial payment of 38.75% of the tranche principal. Here, the running spreads of the first two tranches are 100 bps equally.

In Table 3, for each fixed variance parameters κ_0,κ_1, we calibrate the systematic factor weight $\hat{\rho}$ such that the model quote of equity tranche perfectly matches the market quote of the equity tranche, and we can see "$\Gamma(7)-\Gamma(7)$" simultaneously fits the equity tranche, senior tranche and lower intermediate tranche well. Based on the results of Table 3, we set the variance parameters $\kappa_0=\kappa_1=7$ in Table 4 and calculate the model quote for some specific systematic factor weight ρ. We further get the implied systematic factor weight $\hat{\rho}$ such that model quote equals to market quote for each tranche and find that the implied systematic factor weights $\hat{\rho}$ present a smile shape, that is, the implied systematic factor weights are smaller for intermediate tranches.

Table 5 lists the pricing results for different models. Here, "Gaussian", "Clayton" and "$t(2.1)$-$t(2.1)$" represent the copula among the default times are modeled by one factor Gaussian model, Clayton copula and double t copula with freedom 2.1 equally, respectively. The double t copula model can be expressed as

$$\tau_i=F_i^{-1}\left(H_i\left(Z_i\right)\right),\quad Z_i=\rho\left(\frac{\nu-2}{\nu}\right)^{1/2}Z+\sqrt{1-\rho^2}\left(\frac{\bar{\nu}-2}{\bar{\nu}}\right)^{1/2}\bar{Z}_i,\quad i=1,\dots,n,$$

where $Z,\bar{Z}_i$ are independent random variables following Student t distributions with ν and $\bar{\nu}$ degrees of freedom and $\rho\in[0,1]$, and H_i denotes the distribution function of Z_i. The calibration results for different freedom of double t copula are presented in Table 6, which shows that freedom 2.1 can best fit all tranches simulataneouly. For more details about these three copulas, see Burtschell et al. (2012). The comparison result shows our model outperforms the Gaussian copula and Clayton copula. In addition, the calibration effect of our model is analogous to the double t copula. The two variance parameters of the Gamma processes play the same role as the two freedom parameters of double t copula, which can be used to adjust the left-side and right-side of the distribution of the number of default times just as Figure 2.

Tranches		$[0\%,3\%]$	$[3\%,6\%]$	$[6\%,12\%]$	$[12\%,100\%]$
Market Quotes		38.75%	6.73%	98.83 (bps)	30.7 (bps)
Model Quotes : LFSDC with one factor Gamma Process					
$\Gamma(4)$-$\Gamma(4)$	$\hat{\rho}$=0.8558	38.75%	7.42%	186.49	27.29
$\Gamma(5)$-$\Gamma(5)$	$\hat{\rho}$=0.8668	38.75%	7.26%	179.03	27.91
$\Gamma(6)$-$\Gamma(6)$	$\hat{\rho}$=0.8716	38.75%	7.08%	174.34	28.93
$\Gamma(7)$-$\Gamma(7)$	$\hat{\rho}$=0.8715	38.75%	6.58%	169.68	29.08
$\Gamma(8)$-$\Gamma(8)$	$\hat{\rho}$=0.8747	38.75%	6.44%	169.46	29.29
$\Gamma(9)$-$\Gamma(9)$	$\hat{\rho}$=0.8779	38.75%	6.29%	166.15	29.56
$\Gamma(10)$-$\Gamma(10)$	$\hat{\rho}$=0.8803	38.75%	6.02%	165.50	30.06

Table 3: LFSDC with one factor Gamma process for iTraxx Europe tranches. The prices are quoted by up-front (%) for the first two tranches and spread (bps) for the next two tranches.

Tranches	$[0\%,3\%]$	$[3\%,6\%]$	$[6\%,12\%]$	$[12\%,100\%]$
Market Quotes	38.75%	6.73%	98.83 (bps)	30.7 (bps)
ρ	Model Quotes for $\Gamma(7)$-$\Gamma(7)$			
0.0	84.44%	13.48%	2.78	0
0.2	70.68%	7.81%	88.40	9.0
0.4	61.91%	6.31%	117.14	16.3
0.6	54.25%	5.94%	136.31	22.0
0.8	44.27%	6.18%	156.60	25.6
0.9	36.21%	6.80%	175.44	30.7
Tranche Implied $\hat{\rho}$	0.8715	0.2575	0.2797	0.9000

Table 4: Given $\Gamma(7)$-$\Gamma(7)$, LFSDC with one factor Gamma process for iTraxx Europe tranches. The prices are quoted by up-front (%) for the first two tranches and spread (bps) for the next two tranches.

Tranches	$[0\%,3\%]$	$[3\%,6\%]$	$[6\%,12\%]$	$[12\%,100\%]$
Market Quotes	38.75%	6.73%	98.83 (bps)	30.7 (bps)
$\Gamma(7)$-$\Gamma(7)$, $\hat{\rho} = 0.8715$	38.75%	6.58%	169.68	29.08
Gaussian, $\hat{\rho} = 0.7035$	38.75%	16.11%	242.06	14.53
Clayton, $Kendall\,\hat{\tau} = 0.1508$	38.75%	15.90%	243.22	14.56
$t(2.1)$-$t(2.1)$, $\hat{\rho} = 0.8194$	38.75%	6.53%	141.68	28.49

Table 5: Compare the different models's quotes for iTraxx Europe tranches. The prices are quoted by up-front (%) for the first two tranches and spread (bps) for the next two tranches.

Tranches		[0%, 3%]	[3%, 6%]	[6%, 12%]	[12%, 100%]
Market Quotes		38.75%	6.73%	98.83 (bps)	30.7 (bps)
Model Quotes : Double t copula					
$t\,(2.1)$-$t\,(2.1)$	$\hat{\rho} = 0.8194$	38.75%	6.53%	141.68	28.49
$t\,(2.5)$-$t\,(2.5)$	$\hat{\rho} = 0.7997$	38.75%	7.64%	148.03	26.96
$t\,(3)$-$t\,(3)$	$\hat{\rho} = 0.7850$	38.75%	8.70%	157.64	25.81
$t\,(4)$-$t\,(4)$	$\hat{\rho} = 0.7676$	38.75%	10.38%	173.68	22.95
$t\,(5)$-$t\,(5)$	$\hat{\rho} = 0.7556$	38.75%	11.55%	186.54	21.44

Table 6: Calibration results for double t copula. The prices are quoted by up-front (%) for the first two tranches and spread (bps) for the next two tranches.

Based on the above analysis, the following conclusions can be drawn. Theoretically compared with one factor Gaussian model, this proposed new portfolio credit risk model has sensible financial implications and can incorporate the tail dependence, hence it is capable to capture the extreme default clustering. For CDO tranches pricing, our model outperforms the one factor Gaussian model and produces the similar results to the double t copula. Note that double t copula doesn't have an explicit expression, but the proposed new model has an explicit copula expression. It is known that the tail dependence coefficient of each bivariate marginal copula of double t copula equals to

$$\lambda_{tt}^{L} = \frac{1}{1+\left(\sqrt{1-\rho^2}/\rho\right)^{\nu}} 1_{\{\nu=\bar{\nu}\}} + 1_{\{\nu<\bar{\nu}\}}. \tag{46}$$

Comparing the two equations (44) and (46), it can be seen that double t copula has tail dependence only when $\nu = \bar{\nu}$ and the proposed model is more flexible in modeling tail dependence of defaults. Some additional remarks are as follows.

Remark 5.1. (1) We can directly extend the proposed model to a more general version, i.e., Proposition 4.3 without changing the above model properties.
(2) The factor structure of the time-change processes can be extended to a stochastic correlation version analogous to Burtschel and Gregory (2005), i.e.,

$$T_t^{(i)} = \varsigma_i S_t^{(0)} + (1-\varsigma_i)\, S_t^{(i)}, i = 1, \ldots, n,$$

where $\varsigma_1, \ldots, \varsigma_n$ are independent random variables taking value in $[0,1]$ and $\varsigma_1, \ldots, \varsigma_n$ are independent with $S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(n)}$.

6 Conclusions

This paper first generalizes the notion of the widely-used deterministic distortion function to a random process, named as stochastic distortion. A method for constructing stochastic distortions is provided and some stochastic distortions are given by focusing on some time-change processes. Secondly transforming each component of a copula function by a stochastic distortion, we construct a so-called stochastic distorted copula. Some families of stochastic distorted copulas are given to demonstrate the versatility of stochastic distortions, such as the copula function LFSDC with linear factor property. Finally, LFSDC is applied to model the default correlation in managing portfolio credit risk, and a numeric study shows the advantage of LFSDC over Gaussian copula and double t copula in terms of fitting accuracy and catching tail dependence.

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A Appendix

A.1 Proof of Proposition 2.1

(1) Since $X(u), u \in [0,1]$ is a non-decreasing process and $E[X(u)], u \in [0,1]$ is continuous, we have

$$\lim_{h\to 0^+} P(\mid X(u+h)-X(u)\mid \geq \varepsilon) \leq \lim_{h\to 0^+} \frac{1}{\varepsilon} E\left[\mid X(u+h)-X(u)\mid\right] = \lim_{h\to 0^+} \frac{E(X(u+h)-E(X(u))}{\varepsilon} = 0$$

and $\lim_{h\to 0^-} P(\mid X(u+h)-X(u)\mid \geq \varepsilon) = 0$ as well.

(2) We know that for each $\varepsilon > 0$,

$$\{a \leq X(b+\varepsilon)\} \supseteq \left\{X^{-1}(a) \leq b\right\} \supseteq \{a \leq X(b)\},$$

which implies that

$$P(a \leq X(b+\varepsilon)) \geq P(X^{-1}(a) \leq b) \geq P(a \leq X(b)).$$

Note that when $E[X(u)], u \in [0,1]$ is continuous, the process $X(u), u \in [0,1]$ is stochastic continuous, then

$$P(a \leq X(b+\varepsilon)) - P(a \leq X(b)) \to 0$$

as $\varepsilon \to 0$. Thus we have

$$P(X^{-1}(a) \leq b) = P(a \leq X(b))$$

and

$$P\left(\left\{X^{-1}(a) \leq b\right\} \Delta \{X(b) \geq a\}\right) = 0.$$

(3) We first prove the sufficiency. It follows from the second result that

$$\begin{aligned} P\left(X^{-1}(U) \leq u\right) &= E\left[P\left(X^{-1}(U) \leq u \mid U\right)\right] \\ &= E\left[P\left(U \leq X(u) \mid U\right)\right] \\ &= E\left[P\left(U \leq X(u) \mid X(u)\right)\right] \\ &= E\left[X(u)\right] \\ &= u \end{aligned}$$

holds for $u \in [0,1]$, i.e., $X^{-1}(U)$ equals to U in law.

For the necessity, based on the assumption on $X^{-1}(U)$, for $u \in [0,1]$ we know that $P(X^{-1}(U) = u) = 0$. Since

$$\{X^{-1}(U) \leq u\} \supseteq \{X(u) \geq U\} \supseteq \{X^{-1}(U) < u\},$$

we have

$$u = P\left(X^{-1}(U) \leq u\right) = P\left(X(u) \geq U\right), u \in [0,1],$$

which completes the proof of necessity.

(4) For any convex function ϕ, we have

$$E\left[\phi(U)\right] = E\left[\phi\left(E\left[X(U) \mid U\right]\right)\right] \leq E\left[E\left[\phi\left(X(U)\right) \mid U\right]\right] = E\left[\phi\left(X(U)\right)\right],$$

where the inequality holds because of the Jessen Inequality. Hence $X(U)$ is larger than U in convex order.

A.2 Proof of Theorem 3.1

(1) From Proposition 2.1, for $i = 1, \dots, n$, the variable $X_i^{-1}(U_i)$ is Uniform [0,1] random variable, and

$$\begin{aligned} & P\left(X_1^{-1}(U_1) \le u_1, \dots, X_n^{-1}(U_n) \le u_n\right) \\ = \; & E\left[P\left(X_1^{-1}(U_1) \le u_1, \dots, X_n^{-1}(U_n) \le u_n \mid U_1, \dots, U_n\right)\right] \\ = \; & E\left[P\left(U_1 \le X_1(u_1), \dots, U_n \le X_n(u_n) \mid U_1, \dots, U_n\right)\right] \\ = \; & P\left(U_1 \le X_1(u_1), \dots, U_n \le X_n(u_n)\right) \\ = \; & E\left[B\left(X_1(u_1), \dots, X_n(u_n)\right)\right] = C^{\mathbf{X}|B}(u_1, \dots, u_n), \end{aligned}$$

which implies that the function $C^{\mathbf{X}|B}(u_1, \dots, u_n)$ is a copula, and $\left(X_1^{-1}(U_1), \dots, X_n^{-1}(U_n)\right)$ has distribution $C^{\mathbf{X}|B}(u_1, \dots, u_n)$. The first part of the theorem follows.

(2) Suppose that $(U_1, U_2, \dots, U_n)$ has distribution B_1 and $(V_1, V_2, \dots, V_n)$ has distribution B_2, and the vectors $(U_1, U_2, \dots, U_n)$ and $(V_1, V_2, \dots, V_n)$ are independent of the stochastic distortions $X_1(u), X_2(u), \dots, X_n(u)$, $u \in [0,1]$.

For any $u_1, u_2, \dots, u_n \in [0,1]$, by Proposition 2.1 we have

$$\begin{aligned} & P\left(X_1^{-1}(U_1) \le u_1, X_2^{-1}(U_2) \le u_2, \dots, X_n^{-1}(U_n) \le u_n\right) \\ = \; & P\left(U_1 \le X_1(u_1), U_2 \le X_2(u_2), \dots, U_n \le X_n(u_n)\right) \\ = \; & E\left[P\left(U_1 \le X_1(u_1), U_2 \le X_2(u_2), \dots, U_n \le X_n(u_n) \mid X_1(u_1), X_2(u_2), \dots, X_n(u_n)\right)\right] \\ \le \; & E\left[P\left(V_1 \le X_1(u_1), V_2 \le X_2(u_2), \dots, V_n \le X_n(u_n) \mid X_1(u_1), X_2(u_2), \dots, X_n(u_n)\right)\right] \\ = \; & P\left(V_1 \le X_1(u_1), V_2 \le X_2(u_2), \dots, V_n \le X_n(u_n)\right) \\ = \; & P\left(X_1^{-1}(V_1) \le u_1, X_2^{-1}(V_2) \le u_2, \dots, X_n^{-1}(V_n) \le u_n\right), \end{aligned} \tag{47}$$

where the inequality in (47) comes from the PQD order of B_1, B_2. Thus we have $C^{\mathbf{X}|B_1}(\mathbf{u}) \le C^{\mathbf{X}|B_2}(\mathbf{u})$. Similarly we can prove $\widehat{C^{\mathbf{X}|B_1}}(\mathbf{u}) \le \widehat{C^{\mathbf{X}|B_2}}(\mathbf{u})$. Therefore we have $C^{\mathbf{X}|B_1} \le_{PQD} C^{\mathbf{X}|B_2}$.

(3) The first claim is a direct result of part 2 and the remaining results can be obtained by using the Proposition 7.4 and Corollary 7.1 in Yanagimoto and Okamoto (1969), which show that linear correlation coefficient, Kendall's τ, Spearman's ρ, and Blomquist's q are preserved under PQD order.

A.3 Proof of Proposition 3.2

Using the notation of $V_B([\mathbf{a}, \mathbf{b}])$, we can rewrite the first derivative of the bivariate stochastic distorted copula as

$$\begin{aligned} & \frac{\partial}{\partial u_1} C^{\mathbf{X}|B}(u_1, u_2) \\ = \; & \frac{-1}{l_1(\theta_1)} E\left[\left(\int_0^\infty V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}, T^{(2)}_{\widehat{\omega}_2(u_2)}\right), \mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)} + y_1, T^{(2)}_{\widehat{\omega}_2(u_2)}\right)\right]\right) \nu_1(dy_1)\right)\right], \end{aligned}$$

where $\mathbf{R}(x_1,x_2)=\left(1-e^{\theta_1 x_1},1-e^{\theta_2 x_2}\right)$. Then following the discussion in Section 3.2, we have

$$\begin{aligned}
&\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_1}C^{\mathbf{X}|B}(u_1,u_2)\\
=\;&\lim_{s\to 0}\frac{1}{s}\left[\frac{\partial}{\partial u_1}C^{\mathbf{X}|B}(u_1,u_2+s)-\frac{\partial}{\partial u_1}C^{\mathbf{X}|B}(u_1,u_2)\right]\\
=\;&\lim_{s\to 0}\frac{-1}{sl_1(\theta_1)}E[\int_0^\infty [V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)},T^{(2)}_{\widehat{\omega}_2(u_2+s)}\right),\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1,T^{(2)}_{\widehat{\omega}_2(u_2+s)}\right)\right]\right)\\
&-V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)},T^{(2)}_{\widehat{\omega}_2(u_2)}\right),\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1,T^{(2)}_{\widehat{\omega}_2(u_2)}\right)\right]\right)]\nu_1(dy_1)]\\
=\;&\lim_{s\to 0}\frac{-1}{sl_1(\theta_1)}E\left[\int_0^\infty V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)},T^{(2)}_{\widehat{\omega}_2(u_2)}\right),\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1,T^{(2)}_{\widehat{\omega}_2(u_2+s)}\right)\right]\right)\nu_1(dy_1)\right].
\end{aligned}$$

Furthermore, we can exchange the order of limit and expectation, that is,

$$\begin{aligned}
&\lim_{s\to 0}\frac{1}{s}E\left[\int_0^\infty V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)},T^{(2)}_{\widehat{\omega}_2(u_2)}\right),\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1,T^{(2)}_{\widehat{\omega}_2(u_2+s)}\right)\right]\right)\nu_1(dy_1)\right]\\
=\;&\lim_{s\to 0}\frac{1}{s}E\left[E\left[\int_0^\infty V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)},T^{(2)}_{\widehat{\omega}_2(u_2)}\right),\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1,T^{(2)}_{\widehat{\omega}_2(u_2+s)}\right)\right]\right)\nu_1(dy_1)\mid T^{(1)}_{\cdot}\right]\right]\\
=\;&E\left[\lim_{s\to 0}\frac{1}{s}E\left[\int_0^\infty V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)},T^{(2)}_{\widehat{\omega}_2(u_2)}\right),\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1,T^{(2)}_{\widehat{\omega}_2(u_2+s)}\right)\right]\right)\nu_1(dy_1)\mid T^{(1)}_{\cdot}\right]\right],
\end{aligned}$$

which is due to the fact that

$$\begin{aligned}
&\frac{1}{s}\mid E\left[\int_0^\infty V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)},T^{(2)}_{\widehat{\omega}_2(u_2)}\right),\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1,T^{(2)}_{\widehat{\omega}_2(u_2+s)}\right)\right]\right)\nu_1(dy_1)\mid T^{(1)}_{\cdot}\right]\mid\\
\leq\;&2\frac{1}{s}E\left[\int_0^\infty\mid e^{\theta_2\left(T^{(2)}_{\widehat{\omega}_2(u_2+s)}\right)}-e^{\theta_2\left(T^{(2)}_{\widehat{\omega}_2(u_2)}\right)}\mid\nu_1(dy_1)\right]=2
\end{aligned}$$

and the dominated convergence theorem. We denote

$$h_2\left(x_2;T^{(1)}_{\widehat{\omega}_1(u_1)}\right)=E\left[\int_0^\infty V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)},x_2\right),\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1,x_2\right)\right]\right)\nu_1(dy_1)\mid T^{(1)}_{\cdot}\right],$$

and then we have

$$\begin{aligned}
&\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_1}C^{\mathbf{X}|B}(u_1,u_2)\\
=\;&\frac{-1}{l_1(\theta_1)}E\left[\lim_{s\to 0}\frac{1}{s}E\left[\int_0^\infty V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)},T^{(2)}_{\widehat{\omega}_2(u_2)}\right),\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1,T^{(2)}_{\widehat{\omega}_2(u_2+s)}\right)\right]\right)\nu_1(dy_1)\mid T^{(1)}_{\cdot}\right]\right]\\
=\;&\frac{-1}{l_1(\theta_1)}E\left[\frac{-1}{l_2(\theta_2)}\int_0^\infty\left(h_2\left(T^{(2)}_{\widehat{\omega}_2(u_2)}+y_2;T^{(1)}_{\widehat{\omega}_1(u_1)}\right)-h_2\left(T^{(2)}_{\widehat{\omega}_2(u_2)};T^{(1)}_{\widehat{\omega}_1(u_1)}\right)\right)\nu_2(dy_2)\right]\\
=\;&\frac{1}{l_1(\theta_1)l_2(\theta_2)}E\left[\int_0^\infty\left(h_2\left(T^{(2)}_{\widehat{\omega}_2(u_2)}+y_2;T^{(1)}_{\widehat{\omega}_1(u_1)}\right)-h_2\left(T^{(2)}_{\widehat{\omega}_2(u_2)};T^{(1)}_{\widehat{\omega}_1(u_1)}\right)\right)\nu_2(dy_2)\right]\\
=\;&\frac{1}{l_1(\theta_1)l_2(\theta_2)}E\left[\int_0^\infty\int_0^\infty V_B\left(\left[\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)},T^{(2)}_{\widehat{\omega}_2(u_2)}\right),\mathbf{R}\left(T^{(1)}_{\widehat{\omega}_1(u_1)}+y_1,T^{(2)}_{\widehat{\omega}_2(u_2)}+y_2\right)\right]\right)\nu_1(dy_1)\nu_2(dy_2)\right],
\end{aligned}$$

where equation (16) is applied above for changing the order of the limit and the expectation. The proof is completed.

A.4 Proof of Proposition 4.1

(a) For the first part, by Theorem 3.1 we have

$$\begin{aligned}
& C_{FGM}^{\Pi}(u_1,\ldots,u_n) \\
= & E\left[C_{FGM}\left(\exp\left(\theta_1 T_{\omega_1(u_1)}^{(1)}\right),\ldots,\exp\left(\theta_n T_{\omega_n(u_n)}^{(n)}\right)\right)\right] \\
= & \sum_{k=2}^{n}\sum_{1\le j_1<\cdots<j_k\le n}\eta_{j_1 j_2\cdots j_k}\prod_{i\notin\{j_1,\ldots,j_k\}}E\left[\exp\left(\theta_i T_{\omega_i(u_i)}^{(i)}\right)\right] \\
& \times\prod_{m=1}^{k}E\left[\exp\left(\theta_{j_m}T_{\omega_{j_m}(u_{j_m})}^{(j_m)}\right)\left(1-\exp\left(\theta_{j_m}T_{\omega_{j_m}(u_{j_m})}^{(j_m)}\right)\right)\right]+\prod_{i=1}^{n}E\left[\exp\left(\theta_i T_{\omega_i(u_i)}^{(i)}\right)\right] \\
= & \prod_{i=1}^{n}u_i+\sum_{k=2}^{n}\sum_{1\le j_1<\cdots<j_k\le n}\eta_{j_1 j_2\cdots j_k}\prod_{i\notin\{j_1,\ldots,j_k\}}u_i\prod_{m=1}^{k}\left(u_{j_m}-u_{j_m}^{l(2\theta_{j_m})/l(\theta_{j_m})}\right) \\
= & \prod_{i=1}^{n}u_i\left[1+\sum_{k=2}^{n}\sum_{1\le j_1<\cdots<j_k\le n}\eta_{j_1 j_2\cdots j_k}\prod_{m=1}^{k}\left(1-u_{j_m}^{l(2\theta_{j_m})/l(\theta_{j_m})-1}\right)\right].
\end{aligned}$$

(b) For the second part, first we have

$$\begin{aligned}
& E\left[X^a(u_1)X^b(u_2)\right] \\
= & E\left[\exp\left(\theta a T_{\omega(u_1)}\right)\cdot\exp\left(\theta b T_{\omega(u_2)}\right)\right] \\
= & E\left[\exp\left(\theta(a+b)T_{\omega(u_{(1)})}\right)\cdot\exp\left(\theta b\left(T_{\omega(u_{(2)})}-T_{\omega(u_{(1)})}\right)\right)\right] \\
= & u_{(1)}^{\frac{l(\theta(a+b))-l(\theta b)}{l(\theta)}}u_{(2)}^{\frac{l(\theta b)}{l(\theta)}}.
\end{aligned}$$

By Theorem 3.1, we then get

$$\begin{aligned}
& C_{FGM}^{M}(u_1,u_2) \\
= & E\left[C_{FGM}(X(u_1),X(u_2))\right] \\
= & E\left[X(u_1)X(u_2)+\eta_{12}X(u_1)X(u_2)(1-X(u_1))(1-X(u_2))\right] \\
= & (\eta_{12}+1)u_{(1)}^{\frac{l(2\theta)}{l(\theta)}-1}u_{(2)}+\eta_{12}\left[u_{(1)}^{\frac{l(4\theta)-l(2\theta)}{l(\theta)}}u_{(2)}^{\frac{l(2\theta)}{l(\theta)}}-u_{(1)}^{\frac{l(3\theta)-l(\theta)}{l(\theta)}}u_{(2)}-u_{(1)}^{\frac{l(3\theta)-l(2\theta)}{l(\theta)}}u_{(2)}^{\frac{l(2\theta)}{l(\theta)}}\right].
\end{aligned}$$

A.5 The proof of Proposition 4.3

By Theorem 3.1, we have

$$\Pi_M^{sd}(u_1,\ldots,u_n)=E\left[\prod_{i=1}^{n}\exp\left(\theta_i\sum_{j=0}^{M}a_{i,j}S_{\omega_i(u_i)}^{(j)}\right)\right]=\prod_{j=0}^{M}E\left[\exp\left(\sum_{i=1}^{n}\theta_i a_{i,j}S_{\omega_i(u_i)}^{(j)}\right)\right].$$

Since $\left(u_{\pi(1)},\ldots,u_{\pi(n)}\right)$ is a permutation of $(u_1,\ldots,u_n)$ such that

$$\omega_{\pi(1)}\left(u_{\pi(1)}\right)\le\omega_{\pi(2)}\left(u_{\pi(2)}\right)\le\ldots\le\omega_{\pi(n)}\left(u_{\pi(n)}\right)$$

and $\omega_{\pi(0)}\left(u_{\pi(0)}\right)=0$, we finally obtain

$$\begin{aligned}
&\Pi_M^{sd}\left(u_1,\ldots,u_n\right)\\
=\;&\prod_{j=0}^{M} E\left[\exp\left(\sum_{k=1}^{n}\left(\sum_{i=k}^{n}\theta_{\pi(i)}a_{\pi(i),j}\right)\left(S^{(j)}_{\omega_{\pi(k)}\left(u_{\pi(k)}\right)}-S^{(j)}_{\omega_{\pi(k-1)}\left(u_{\pi(k-1)}\right)}\right)\right)\right]\\
=\;&\prod_{j=0}^{M}\exp\left(\sum_{k=1}^{n}\left(\omega_{\pi(k)}\left(u_{\pi(k)}\right)-\omega_{\pi(k-1)}\left(u_{\pi(k-1)}\right)\right)l_j\left(\sum_{i=k}^{n}\theta_{\pi(i)}a_{\pi(i),j}\right)\right)\\
=\;&\exp\left(\sum_{j=0}^{M}\sum_{k=1}^{n}\left(\omega_{\pi(k)}\left(u_{\pi(k)}\right)-\omega_{\pi(k-1)}\left(u_{\pi(k-1)}\right)\right)l_j\left(\sum_{i=k}^{n}\theta_{\pi(i)}a_{\pi(i),j}\right)\right).
\end{aligned}$$

A.6 CDO pricing formula with detailed results and data

We first give the detailed formula for CDO tranche pricing according to Hull and White (2004). Let $L(s)$ be the cumulative loss on the CDO portfolio at time s. If each component of the CDO portfolio has the same nominal FV and recover rate REC, then $L(s)$ can be expressed as

$$L(s)=FV\cdot(1-REC)\cdot N(s),$$

where $N(s)=\sum_{i=1}^{n}1_{\{\tau_i\leq s\}}$ is the number of defaults in the credit portfolio until time s. Consider a tranche with a lower threshold K_1 and upper threshold K_2. Let $M(s)$ be the cumulative loss of the tranche, i.e.,

$$M(s)=\min\left(\max(L(s)-K_1,0),K_2-K_1\right).$$

Suppose that $0=T_0<T_1<\cdots<T_K<T_M$ are the premium payment dates and T_0,T_M are the initial date and maturity date, respectively. Then we can express the price of the default leg of the given tranche as

$$Default\,leg=E\left[\sum_{k=0}^{K}D\left(0,T_k\right)\cdot\left(M\left(T_{k+1}\right)-M\left(T_k\right)\right)\right],$$

where $M(0)=0$, $T_{K+1}=T_M$ and $D(0,t)$ represents the discount factor. When the CDO tranche price is the up-front quote, the premium leg can be expressed as

$$Premium\,leg=RuningSpread\cdot E[\sum_{k=0}^{K}D\left(0,T_k\right)\cdot\max\left(K_2-K_1-M\left(T_k\right),0\right)]+Up\text{-}front\cdot\left(K_2-K_1\right),$$

where running spread is a fixed premium, e.g. 100bp for iTraxx first two tranches. Here up-front is quoted by percentage such that $Premium\,leg=Default\,leg$. When the CDO price is conventionally quoted by premium, we have

$$Premium\,leg=Premium\cdot E[\sum_{k=0}^{K}D\left(0,T_k\right)\cdot\max\left(K_2-K_1-M\left(T_k\right),0\right)],$$

where the premium is determined at the initial time such that $Premium\,leg=Default\,leg$. The tranche premium is used to compensate the default leg of the given tranche, that is, the higher default leg the higher premium.

The following two tables list the interest rate curve and CDS spreads of CDO's components used in Section 5.

1M	2M	3M	6M	1Y	2Y	3Y	4Y	5Y
-0.342	-0.288	-0.249	-0.143	-0.011	-0.151	-0.116	-0.045	0.048

Table 7: Europe standard swap rate curve. The unit is %.

Name	1Y	2Y	3Y	4Y	5Y	Name	1Y	2Y	3Y	4Y	5Y
1	19.68	26.63	33.72	44.69	51.38	39	91.13	96.74	101.98	117.46	126.88
2	28.26	45.55	62.26	81	98.62	40	20.77	29.86	39.51	50.86	62
3	26.65	36.01	46.25	61.32	70.5	41	9.13	17.46	25.7	35.51	45.5
4	26.63	32.9	39.15	46.96	53.97	42	17.63	25.57	35.24	47.53	60.33
5	15.85	25.05	33.99	43.81	54.29	43	135.98	140.43	140.99	148.99	153.83
6	16.24	20.05	24.67	32.03	36.55	44	15.63	21.92	28.05	36.9	46
7	87.59	213.85	309.82	387.13	497.91	45	10.91	17.86	25.29	33.42	42.19
8	25.68	34.76	43.83	55.94	67.5	46	25.79	40.35	56.11	74.65	93.6
9	45.51	58.65	71.96	87.52	101.21	47	18.67	32.64	46.65	62.6	72.3
10	9.03	14.97	22.64	28.76	36	48	15.76	26.64	39.18	50.65	62.2
11	21.44	31.99	43.55	53.68	64.26	49	12.31	18.83	27.04	37.95	48.7
12	12.95	19.59	28.49	39.64	50.39	50	16.9	32.62	46.49	63.81	79.8
13	35.19	45.63	56.1	70.59	79.41	51	29.81	36.49	43	51.21	59.49
14	25.17	38.61	51.81	64.04	76.5	52	26.42	39.19	50.94	73.47	91.4
15	12.14	16.48	21.98	30.19	38.5	53	9.16	13.74	20.47	27.33	34.5
16	26.2	35.74	43.66	60.97	71.5	54	15.05	27.82	39.75	55.5	71.4
17	23.61	31.98	50.57	67.63	82.8	55	17.37	33.49	50.38	66.33	80.4
18	45.17	59.14	73.61	97.82	112.52	56	87.83	157.01	216.81	278.64	353.33
19	46.14	60.95	76.36	99.5	113.56	57	54.44	60.13	67.24	81.08	89.5
20	71.58	79.4	87.32	100.5	108.51	58	10.61	17.77	23.72	32.95	38.6
21	14.47	19.4	25.19	31.73	38.29	59	26.32	32.29	39.07	52.7	61
22	20.95	30.12	40.28	50.96	61.5	60	36.65	53.81	76.24	99.08	121
23	22.18	33.28	42.88	54.56	65	61	18.88	29.95	41.91	55.29	69
24	4.67	9.64	15.19	24.67	34	62	51.57	67.77	82	100.66	112
25	12.14	22.43	32.23	42.42	53.79	63	20.65	31.19	42.48	54.72	66.6
26	12.05	20.32	30.35	40.65	50.5	64	11.3	18.39	26.05	37.67	49.5
27	25.66	39.05	53.22	68.72	82.21	65	8.56	13.89	19.78	27.4	35.5
28	16.52	30.92	44.98	59.84	74.5	66	17.1	26.62	39.76	53.56	67.2
29	19.28	30.32	40.61	53.83	66.5	67	10.02	16.89	24.3	33.77	43.33
30	112.68	161.85	200.18	236.97	259.26	68	8.34	22.47	36.78	52.26	68.1
31	34.97	48.26	64.09	79.96	96.8	69	10.86	16.3	23.56	30.83	38.5
32	13.97	21.46	30.6	41.01	51.79	70	24.06	43.32	63.17	78.96	96.1
33	16.35	26.1	35.13	47.1	59.2	71	8.08	13.95	20.38	26.62	32.68
34	57.12	68.65	80.11	93.67	101.9	72	42.58	57.15	70.68	82.48	89.66
35	8.28	12.56	17	22.89	29.35	73	36.31	57.72	79.92	108.11	132.2
36	16.41	23.27	33.09	45.75	59.9	74	98.29	117.15	133.62	150.04	160
37	19.36	29.99	40.42	52.9	60.5	75	11.44	15.62	20.89	30.53	36.4
38	24.73	33.78	43.17	60.78	71.5	76	12.52	17.74	24.86	33.07	41.8

Name	1Y	2Y	3Y	4Y	5Y	Name	1Y	2Y	3Y	4Y	5Y
77	6.77	11.68	16.06	20.49	25.5	102	30	48.26	71.94	93.67	115.5
78	31.39	47.72	62.82	78.71	95	103	14.41	21.74	32.32	44.1	55.6
79	17.24	29.42	41.4	52.28	66.24	104	16.11	27.08	38.53	53.22	65.5
80	32.21	48.38	65.36	85.53	105	105	24.15	36.28	49.87	67.39	85
81	14.73	25.05	36.22	49.46	64.1	106	33.56	52.22	72.88	95.08	117
82	13.64	22.93	32.56	44.76	57.65	107	15.67	27.87	37.96	52.42	66.94
83	17.26	29.48	44.21	58.96	74.17	108	17.79	25.49	33.69	42.67	51.7
84	10.41	17.45	24.77	33.88	43	109	15.2	22.98	31.43	42.35	53.1
85	38.83	57.22	77.21	100.34	122.8	110	34.17	40.93	47.14	54.46	61.13
86	17.92	27.01	35.43	47.48	58.29	111	84.58	101.78	116.17	138.05	151.34
87	75.7	98.82	140.01	177.85	211.33	112	5.57	10.39	14.48	19.36	24.5
88	32.6	56.26	84.87	104.81	123.24	113	9.75	19.78	34.63	53.86	72.6
89	68.83	77.07	82.81	98.9	108.67	114	16.66	29.95	42.32	58.35	74.61
90	16.43	25.1	33.74	50.07	64.39	115	12.49	21.64	31.62	41.91	52.19
91	7.59	11.9	16.25	19.89	24.5	116	18.26	26.99	37.65	48.31	59.49
92	25.02	42.53	59.97	83.24	106	117	9.66	18.31	26.1	34.72	43.78
93	6.66	9.75	14.99	24.55	30.4	118	16.49	26.26	35.99	47.5	59.5
94	8.97	15.88	22.97	30.56	39	119	24.11	39.61	55.96	74.09	92.49
95	25.78	37.64	52.72	72.25	90.67	120	65.29	85.96	106.64	122.61	140
96	27.92	36.71	43.17	60.78	71.5	121	25.92	41.9	60.94	79.97	99
97	23.38	39.9	57.96	75.01	91.7	122	15.13	21.81	29.37	40.87	53
98	70.03	80.68	91	114.95	129.49	123	23.31	43.68	66.47	94.7	122.5
99	14.32	22.87	31.4	44.68	55.8	124	7.94	12.98	19.33	26.85	34.5
100	18.08	33.26	48.66	65.21	82.19	125	11.21	18.58	24.31	33.45	39.03
101	11.52	15.84	21.39	31.43	37.57						

Table 8: Marginal CDS spread for each component of iTraxx Europe S24. The unit is basis point.