Theory of Coordinated Agency

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Outline

Outline

Science, Technology, and Mathematics

Natural science is an expansion of observing; technology, of contriving; mathematics, of understanding.

— Michael Polanyi
Personal Knowledge

- Economics and engineering are different sides of the same coin
  - Economists use mathematical models to analyze reality
  - Engineers use mathematical models to synthesize reality
  - The two sides constantly overlap

Analysis uses models to reduce reality *to* an abstraction, while synthesis uses models to create a reality *from* an abstraction.

The purpose of abstraction is not *to be vague, but to create a new semantic level in which one can be absolutely precise.*

— Edsger Dijkstra
ACM Turing Lecture, 1972

Increasing Model Precision

- Interdisciplinary study offers different perspectives on the same fundamental issue
- Viewing phenomena from a different perspective can lead to more precision
- A phenomenon that is critical to the study of multiagent behavior is *coordination*
  - Etymology: Latin: *co* (together) + *ordinare* (to regulate).
  - Oxford English Dictionary:

  To place or arrange (things) in proper position relative to each other and to the system of which they form parts; to bring into proper combined order as parts of a whole.

- Goal: an explication of *coordinate*

Coordination vs Cooperation

- Cooperation deals with the commonality of individual objectives
  - Performance is expressed operationally via individual payoffs
  - No notion of systematic group-level behavior need be relevant
  - Cooperation is intrinsically an individual-level phenomenon
- Coordination deals with the functionality of a group of individuals as they interact and produce systematic group-level behavior
  - Cooperative coordination (e.g., teamwork)
  - Conflictive coordination (e.g., athletic contests)
  - Mixed coordination (e.g., families)
  - Coordination is intrinsically a group-level phenomenon
- Question: Can coordination be operationally quantified?

Extant Solution Concepts for Coordination

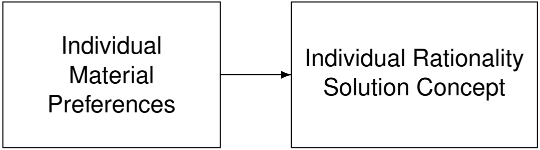
- Schelling (1960) — tacit coordination: "Most situations . . . provide some clue for coordinating behavior, some focal point for each person's expectation of what the other expects him to expect to be expected to do."
- Lewis (1969) — coordination by convention
- Bicchieri (1993) — learned coordination as the end product of social evolution
- Cooper (1999) — technological complementarity
- Bacharach (2006) — agent and utility transforms
- Sugden (2014) — individuals play their parts in the interest of mutual benefit

These approaches all have at least one thing in common: The division of labor between the specification of the preferences and the specification of the solution concept

The Division of Labor Model

Economic theory proceeds largely to take wants as fixed. This is primarily a division of labor. The economist has little to say about the formation of wants; this is the province of the psychologist. The economist's task is to trace the consequences of any given set of wants.

— Milton Friedman
Price Theory



- Preferences are fixed and immutable — *categorical*
- Preferences are independent of social context

Addressing the Limits of Division of Labor

- Categorical preference models express individual material benefit
- Solution concepts defines individually rational behavior
  - Individual rationality based solutions (e.g., equilibrium) match the structure of the categorical preference model
  - Coordinated solutions do not match the structure of the categorical preference model, because coordination involves social context that extends beyond narrow self-interest
- Coordination-based solution concepts, however, are actually manifestations of preference — social preference
- This conflates preferences and solution concepts
- The key issue: how to incorporate social preferences into a game?

Modeling Social Preference as Material Preference

- Defining social preference in terms of material preference is problematic

We model fairness as self-centered inequity aversion. Inequity aversion means that people resist inequitable outcomes; i.e., they are willing to give up some material payoff to move in the direction of more equitable outcomes. Inequity aversion is self-centered if people do not care per se about inequity that exists among other people but are only interested in the fairness of their own material payoff relative to the payoff of others.

— E. Fehr and K. Schmidt
Quarterly Journal of Economics

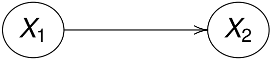
- Fairness is an inherently social attribute that is not easily simulated with a framework designed to express material benefit

A Framework for Coordinated Agency

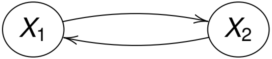
- Agents incorporate their sensitivity to the preferences of others into their own individual preferences
- As the social event is engaged (i.e., the game is played), this sensitivity propagates through the collective, forming social relationships as the players interact
- Social interaction generates an emergent group-level phenomenon
- To the extent that this phenomenon generates systematic group-level behavior, some concept of coordination may emerge
- The nature of the coordination is difficult to predict *ex ante* and may not be expressible in terms of individual payoffs

Influence Propagation Models

- Acyclic models: influence flows unidirectionally:

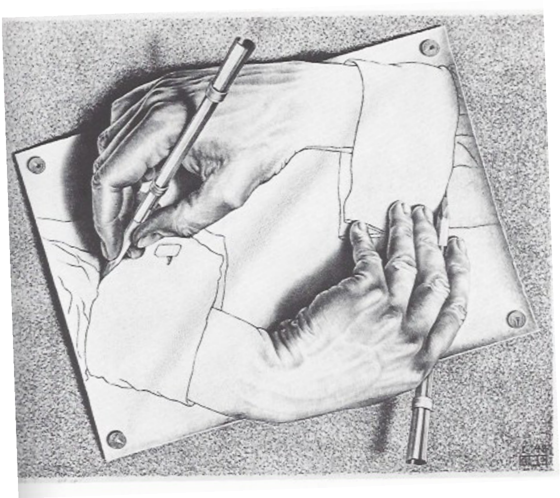


  - An appropriate model for hierarchical groups
  - We will examine this case first
- Cyclic models: influence flows bidirectionally



  - Enables reciprocity
  - Requires a more complicated analysis

The Emergence of Coordination à la Escher



Outline

Classical Noncooperative Normal-Form Game Model

- *Agents* (players): $\{X_1, \ldots, X_n\}$
- Each X_i possesses a finite *action set* $\mathcal{A}_i = \{z_{i1}, \ldots, z_{iN_i}\}$
- *Outcome set*: $\boldsymbol{\mathcal{A}} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$
- *Action profile*: $\mathbf{a} = (a_1, \ldots, a_n)$, with $a_i \in \mathcal{A}_i$
- *Utilities* (payoffs): u_i: $\boldsymbol{\mathcal{A}} \to \mathbb{R}$
- Payoff array: An $N_1 \times \cdots \times N_n$ dimensional structure such that the $(k_1, \ldots, k_n)$th entry is the sub-array

$$[u_1(z_{ik_1}, \ldots, z_{nk_n}), \ldots, u_n(z_{ik_1}, \ldots, z_{nk_n})]$$

  where $u_i(z_{ik_1}, \ldots, z_{nk_n})$ is the payoff that X_i receives if the profile $(z_{ik_1}, \ldots, z_{nk_n})$ is actualized
- When $n = 2$, the payoff array is termed a *payoff matrix*

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Expressing Preference via Hypothetical Propositions

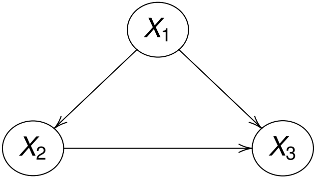
- *Conditionalization*: a concept from Bayesian epistemology
- Form an analogy with probability theory
  - Agents X_1, X_2 are analogous to random variables Y_1 and Y_2
  - Relationships are expressed via hypothetical propositions
- Epistemological context: *conditional belief*
  - Antecedent: The event y_1 is realized by Y_1, written $Y_1 = y_1$
  - Consequent: $Y_2 = y_2$ as governed by the conditional probability $Prob(Y_2 = y_2 | Y_1 = y_1)$
- Social context: *conditional preference*
  - Antecedent: X_1 intends outcome $\mathbf{a}_1$, written as $X_1 \models \mathbf{a}_1$
  - Consequent: X_2 intends outcome $\mathbf{a}_2$ as governed by the conditional preference $Pref(X_2 \models \mathbf{a}_2 | X_1 \models \mathbf{a}_1)$

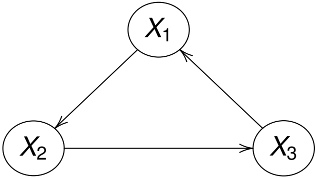
For additional details see Stirling (2012); Stirling and Felin (2013)

Modeling a Society as an Influence Network

A *network* $\{X_1, \ldots, X_n\}$ is a graph whose vertices are agents and whose edges express social influence relationships between agents

- Edges are *directed*, written $X_j \rightarrow X_i$, if influence is unidirectional from X_j to X_i
- A *directed path* of length k from X_{i_1} to X_{i_k} is a sequence $\{X_{i_1}, \ldots, X_{i_k}\}$ such that an $X_{i_j} \rightarrow X_{i_{j+1}}$, $j = 1 \ldots, k-1$, written $X_j \mapsto X_{i_k}$
- A path is a *cycle*, or *closed path*, if $X_j \mapsto X_j$
- A directed graph is *acyclic* if there are no cycles





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The Conditional Game Network

- Let $\{X_1, \ldots, X_n\}$ denote an collective of agents, with each X_i possessing a finite action set $\mathcal{A}_i$, $i = 1, \ldots, n$
- Let $\boldsymbol{\mathcal{A}} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ denote the set of *action profiles*
- A *conjecture* for X_i, denoted a_{ii}, is a hypothesized intention that X_i will actualize a_{ii}
- A *conjecture profile*, denoted $\mathbf{a}_i = (a_{i1}, \ldots, a_{ii}, \ldots, a_{in}) \in \boldsymbol{\mathcal{A}}$, is a set of hypothesized intentions by X_i that $\{X_1, \ldots, X_n\}$ will actualize $\mathbf{a}_i$
- The *parents* of X_i, denoted $\mathrm{pa}(X_i) = \{X_{i_1}, \ldots, X_{i_{p_i}}\}$, is the subset of $\{X_1, \ldots, X_n\}$ that directly influences X_i
- A *conditioning conjecture* for X_i, denoted $\boldsymbol{\alpha}_{\mathrm{pa}(i)} = (\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_{p_i}})$, is a set of conjecture profiles for $\mathrm{pa}(X_i)$
- A *conditional utility* for X_i, denoted $u_{i|\mathrm{pa}(i)}(\cdot|\boldsymbol{\alpha}_{\mathrm{pa}(i)}) \colon \boldsymbol{\mathcal{A}} \to \mathbb{R}$, defines an ordering over the outcomes conditioned on the conjectures for $\mathrm{pa}(X_i)$

Coordination via Conditional Game Theory

- Build on the analogy with Bayesian epistemology
  - Normalize all utilities to become *mass functions*

$$u_{i|\mathrm{pa}(i)}(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)}) \geq 0 \;\forall \boldsymbol{\alpha}_{\mathrm{pa}(i)} \in \boldsymbol{\mathcal{A}}^{p_i}$$

$$\sum_{\mathbf{a}_i} u_{i|\mathrm{pa}(i)}(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)}) = 1 \;\forall \boldsymbol{\alpha}_{\mathrm{pa}(i)} \in \boldsymbol{\mathcal{A}}^{p_i}$$

- The resulting network is *isomorphic* to a Bayesian network
  - Agents are isomorphic to random vectors (multivariate random variables)
  - Conditional utilities are isomorphic to conditional probabilities

Bayesian Networks

A *Bayesian network* is a directed acyclic graph that satisfies the following conditions.

- The ith vertex corresponds to a discrete random variable Y_i taking values in a finite set $\mathcal{Y}_i$.
- The ith edge corresponds to the conditional probability that $Y_i = y_i \in \mathcal{Y}_i$, given that its parents $\mathrm{pa}\,(Y_i) = \{Y_{i_1}, \ldots, Y_{i_{q_i}}\}$ assume the values $\mathbf{y}_i = (y_{i_1}, \ldots, y_{i_{q_i}}) \in \mathcal{Y}_{i_1} \times \cdots \times \mathcal{Y}_{i_{q_i}}$, is denoted by $p_{i|\mathrm{pa}(i)}(y_i|\mathbf{y}_i)$. If $\mathrm{pa}\,(Y_i) = \emptyset$, then $p_{i|\mathrm{pa}(i)} = p_i$, the unconditional marginal distribution of Y_i.

For more details, see Pearl (1988); Jensen (2001); Lauritzen (1996)

The Fundamental Theorem of Bayesian Networks

Theorem

Let $\{Y_1, \ldots, Y_n\}$ be a Bayesian Network with conditional probability mass functions for each vertex given by $p_{i|\mathrm{pa}(i)}(y_i|\mathbf{y}_i)$, where $\mathbf{y}_i = (y_{i_1}, \ldots, y_{i_{q_i}})$ are the realizations of $\mathrm{pa}\,(Y_i) = \{Y_{i_1}, \ldots, Y_{i_{q_i}}\}$, the parents of Y_i. If $\mathrm{pa}\,(Y_i) = \varnothing$, then $p_{i|\mathrm{pa}(i)} = p_i$, an unconditional mass function. Then

$$p_{1:n}(y_1, \ldots, y_n) = \prod_{i=1}^{n} p_{i|\mathrm{pa}(i)}(y_i|\mathbf{y}_i)$$

is the unique joint probability mass function for $\{Y_1, \ldots, Y_n\}$

The joint probability mass function incorporates all of the statistical dependencies that exist among the random phenomena — a comprehensive statistical model

The Coordination Function

Let $\{X_1, \ldots, X_n\}$ be a directed acyclic influence network with conditional utility mass functions for each agent given by $u_{i|\mathrm{pa}(i)}(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)})$, where $\boldsymbol{\alpha}_{\mathrm{pa}(i)} = (\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_{q_i}})$ is the conditioning conjecture for $\mathrm{pa}\,(X_i) = \{X_{i_1}, \ldots, X_{i_{p_i}}\}$. If $\mathrm{pa}\,(X_i) = \varnothing$, then $u_{i|\mathrm{pa}(i)} = u_i$, an unconditional mass function. Then

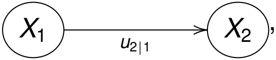
$$u_{1:n}(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \prod_{i=1}^{n} u_{i|\mathrm{pa}(i)}(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)})$$

is the *coordination function* for $\{X_1, \ldots, X_n\}$

The coordination function incorporates all of the social influence relationships that exist among the agents — a comprehensive social model

Coordination as Preference Connections

Consider the 2×2 network



, where X_1 possesses a categorical utility u_1 and X_2 possesses a conditional utility $u_{2|1}$.

- u_1 expresses strict individual preference
- $u_{2|1}$ expresses an expanded notion of individual preference, whereby X_2 takes into account the influence exerted by X_1
- Combining u_1 and $u_{2|1}$ defines preference connectivity:

$$u_{12}[(a_{11}, a_{12}), (a_{21}, a_{22})] = u_1(a_{11}, a_{12}) u_{2|1}(a_{21}, a_{22} | a_{11}, a_{12})$$

- Preference connectivity is *not* with respect to material payoffs — no such entities are defined by the game
- Preference connectivity is *not* joint preference in the sense of subsuming individual preferences into a group preference — preference remains an individual concept

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Ex Post Marginal Utilities

- The coordination function is a function of n^2 variables — a complete conjecture profile for each agent
- The *ex post marginal utility* for X_i is

$$v_i(\mathbf{a}_i) = \sum_{\sim \mathbf{a}_i} u_{1:n}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$$

- $v_i(\mathbf{a}_i)$ expresses X_i's unconditional utility over $\mathcal{A}$ after taking into consideration all social influence
- These utilities can be juxtaposed into a payoff array and solved via classical means (e.g.„ Nash equilibria)
- Such solution concepts, however, are still based on narrow self interest
- But this is not the end of the story —there is more to be said

The Coordination Utility

- A conjecture profile $\mathbf{a}_i = (a_{i1}, \ldots, a_{in})$ is a hypothesis by X_i that each X_j will actualize a_{ij}, $j = 1, \ldots, n$
- Each X_i, however, has control only over a_{ii}, its own component
- What is most relevant regarding group behavior is how the individual conjectures a_{ii}, $i = 1, \ldots, n$, fit together
- The *coordination utility* is the marginal of the coordination function with respect to each individual's component of its conjecture profile

$$w_{1:n}(a_{11}, \ldots, a_{nn}) = \sum_{\sim a_{11}} \cdots \sum_{\sim a_{nn}} u_{1:n}[(a_{11}, \ldots, a_{1n}), \ldots, (a_{n1} \ldots, a_{nn})]$$

  - Quantizes the degree to which the agents fit together to function systematically
  - A measure of ecological compatibility

Coordinated Decisions

- Coordinated Individual RationalIty
  - The *coordinated individual decision function* for X_i is the ith marginal of the coordination utility

$$w_i(a_{ii}) = \sum_{\sim a_{ii}} w_{1:n}(a_{11}, \ldots, a_{1n})$$

  - The *coordinated individually rational decision* for X_i is $a_{ii}^* = \arg\max_{a_{ii} \in \mathcal{A}_i} w_[(a_{ii})$
- Generalized Team Reasoning (*GTR*):
  - The *GTR profile* is

$$\mathbf{a}^\dagger = \arg\max_{\mathbf{a} \in \boldsymbol{\mathcal{A}}} w_{1:n}(\mathbf{a})$$

  - The generalized team reasoning decision for X_i is $a_{ii}^\dagger$, its component of the *GTR* profile $\mathbf{a}^\dagger = (a_{11}^\dagger, \ldots, a_{nn}^\dagger)$
- A *consensus* obtains if $a_{ii}^* = a_{ii}^\dagger$, $i = 1, \ldots, n$
- A consensus is not guaranteed: players have clear choices — either adopt individual rationality or team reasoning

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Additional Motivation for this Structure — Coherence

- X_i is *subjugated* if, whenever it most prefers a_{ii}, then

$$w_{1:n}(a_{11}, \ldots, a_{ii}, \ldots, a_{nn}) < w_{1:n}(a_{11}, \ldots, a'_{ii}, \ldots, a_{nn})$$

  for all $a'_{ii} \neq a_{il}$ and for all $(a_{11}, \ldots, a_{i-1\,i-1}, a_{i+1],i+1}, \ldots, a_{nn})$ What ever action X_i deems of most value to it, any outcome that contains that action minimizes coordination
- Subjugation is isomorphic to a gamble resulting in a sure loss
- The Dutch Book theorem: sure loss is impossible if, and only if, bets are placed according to the probability syntax (Ramsey, 1950; de Finetti, 1937; Kemeny, 1955; Lehman, 1955)
- *Coherence:* subjugation is impossible if, and only if, utilities comply with the syntax of probability theory
- Coherence is perhaps the weakest possible concept of democracy

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Model Simplifications

Conditional game theory enables model simplifications that are not possible with conventional game theory

- X_i is *dissociated* if its conditional utility is a function of, and only of, its own individual conjecture and the individual conjectures of all of its parents

$$u_{i|\mathrm{pa}(i)}(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)}) = u_{i|\mathrm{pa}(i)}(a_{ii}|a_{i_1 i_1} \dots a_{i_{p_i} i_{p_i}}) \tag{1}$$

- The coordination function for a fully dissociated network is

$$u_{1:n}(a_{11}, \dots, a_{nn}) = \prod_{i=1}^{n} u_{i|\mathrm{pa}(i)}(a_{ii}|a_{i_1 i_1} \dots a_{i_{p_i} i_{p_i}})$$

- Notice that, for this special case,

$$w_{1:n}(a_{11}, \dots, a_{nn}) = u_{1:n}(a_{11}, \dots, a_{nn})$$

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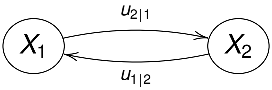
Outline

Extension to Networks with Cycles

- As it stands, this approach applies only to acyclic networks
  - Feedback not permitted
  - Restricted to hierarchical networks
- Many networks involve feedback

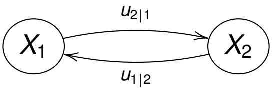


- X_1 influences X_2 who in turn influences X_1 and so on
- Problem: Can lead to an infinite regress

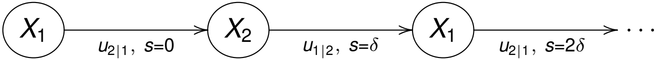
Modeling Reciprocity

- View the 2×2 dissociated cyclic network with

$$\mathcal{A}_1 = \{z_{11}, z_{12}\} \qquad \mathcal{A}_2 = \{z_{21}, z_{22}\}$$

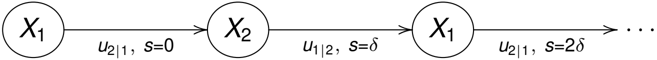


as a time sequence of acyclic networks



- Key issue: Convergence

Transition Dynamics



- Initialization: at time $s = 0$, suppose X_1's individual decision function is $w_1(a_{11}, 0)$
- At time $s = \delta$, the coordination utility is

$$w_{12}(a_{11}, a_{22}, \delta) = u_{2|1}(a_{22}|a_{11})w_1(a_{11}, 0)$$

and X_2's individual decision function is

$$w_2(a_{22}, \delta) = \sum_{a_{11}} w_{12}(a_{11}, a_{22}, \delta)$$

- at time $s = 2\delta$, the coordination utility is

$$w_{12}(a_{11}, a_{22}, 2\delta) = u_{1|2}(a_{11}|a_{22})w_2(a_{22}, \delta)$$

and X_1's individual decision function is

$$w_1(a_{11}, 2\delta) = \sum_{a_{22}} w_{12}(a_{11}, a_{22}, 2\delta)$$

Transition Dynamics — Matrix Formulation

- Define the state-to-state transition matrices

$$T_{1|2} = \begin{bmatrix} u_{1|2}(z_{11}|z_{21}) & u_{1|2}(z_{11}|z_{22}) \\ u_{1|2}(z_{12}|z_{21}) & u_{1|2}(z_{12}|z_{22}) \end{bmatrix}$$

$$T_{2|1} = \begin{bmatrix} u_{2|1}(z_{21}|z_{11}) & u_{2|1}(z_{21}|z_{12}) \\ u_{2|1}(z_{22}|z_{11}) & u_{2|1}(z_{22}|z_{12}) \end{bmatrix}$$

- Define the *individual decision vector*

$$\mathbf{w}_1(s) = \begin{bmatrix} w_1(z_{11}, s) \\ w_1(z_{12}, s) \end{bmatrix}$$

$$\mathbf{w}_2(s) = \begin{bmatrix} w_2(z_{21}, s) \\ w_2(z_{22}, s) \end{bmatrix}$$

- Then

$$\mathbf{w}_1(s + 2\delta) = T_{1|2}\mathbf{w}_2(s + \delta) = T_{1|2}T_{2|1}\mathbf{w}_1(s)$$
$$\mathbf{w}_2(s + 2\delta) = T_{2|1}\mathbf{w}_1(s + \delta) = T_{2|1}T_{1|2}\mathbf{w}_2(s)$$

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Closed-Loop Transition Matrix

- Define the closed-loop transition matrices

$$T_1 = T_{1|2} T_{2|1}$$
$$T_2 = T_{2|1} T_{1|2}$$

- Let the time index t be measured in increments of 2δ
- Then

$$\mathbf{w}_i(t) = T_i \mathbf{w}_i(t-1) = T_i T_i \mathbf{w}_i(t-2) = \cdots = T_i^t \mathbf{w}_i(0)$$

- Question: Does

$$\overline{\mathbf{w}}_i = \lim_{t\to\infty} \mathbf{w}_i(t) = \lim_{t\to\infty} T_i^t \mathbf{w}_i(0)$$

exist?

Markov Convergence Theorem

Theorem

If T_i^m is strictly positive for some finite m and all columns sum to unity, then

- *T_i possesses a unique unity eigenvalue (Frobenius-Perron Theorem)*
- *Let $\overline{\mathbf{w}}_i$ denote the corresponding eigenvector*
- $\lim_{t\to\infty} T_i^t = \begin{bmatrix} \overline{\mathbf{w}}_i & \cdots & \overline{\mathbf{w}}_i \end{bmatrix}$
- $\overline{\mathbf{w}}_i = \lim_{t\to\infty} T_i^t \mathbf{w}_i(0)$ *for all* $\mathbf{w}_i(0)$
- *Convergence is exponentially fast (Doob)*

$\overline{\mathbf{w}}_i$ is X_i's *steady-state* individual decision vector

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Interpretation of Convergence

- Virtual bargaining

  . . . agreements that the social participants anticipate they would make, were they to engage in explicit bargaining . . . [and] operates within the framework of rational-choice theory . . . [by] extend[ing] the scope of rational-choice models of interaction

  — Misyak, Melkonyan, Zeitoun, Chater
  Trends in Cognitive Sciences

- Direct negotiations if possible

Outline

A Coordination Index

- The coordination utility $w_{1:n}$ establishes an ordering over action profiles with regard to the degree of coordination
- The individual decision function w_i establishes an ordering over individual action sets with regard to individual performance
- Key issue: Is it possible to quantify the degree to which the interests of the agents are connected due to social influence?
  - Cooperative coordination
  - Conflictive coordination
  - Mixed coordination
- Ecological compatibility: How fit are the agents to function in their environment?

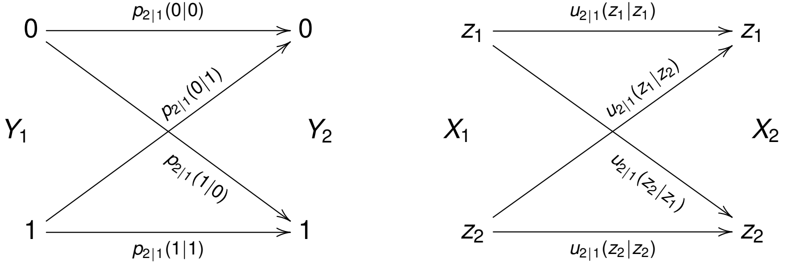
Characterizing Coordination Mathematically

- Consider the following questions:
  - How predictable is the realization of a random phenomenon?
  - Given knowledge of the realization of one random phenomenon, how predictable is the realization of another random phenomenon?
- Because of the isomorphism between conditional utility theory and probability theory, these questions are analogous to the following:
  - How predictable is an agent's behavior?
  - Given knowledge of one agent's behavior, how predictable is another agent's behavior?

Shannon Information Theory

Shannon information theory provides a mathematical characterization of a communications system

- A transmitter encodes a message into a sequence of *bi*nary digi*ts* or *bits* (0's and 1's)
- A receiver decodes the signal into the message
- An error occurs if an encoded 0 is decoded as a 1 or vice versa
- The social influence scenario is isomorphic



Entropy and Mutual Information

- *Individual entropy:* the average behavioral uncertainty for X_i

$$H(X_i) = -\sum_{a_{ii}} w_i(a_{ii}) \log_2 w_i(a_{ii})$$

- *Group entropy:* the average behavioral uncertainty for the group

$$H(X_1, \ldots, X_n) = -\sum_{a_{11}} \cdots \sum_{a_{nn}} w_{1:n}(a_{11}, \ldots a_{nn}) \log_2 w_{1:n}(a_{11}, \ldots a_{nn})$$

- *Mutual information:* the difference between the sum of the average individual behavioral uncertainties and the behavioral uncertainty of the group

$$I(X_1, \ldots, X_n) = \sum_{i=1}^{n} H(X_i) - H(X_1, \ldots, X_n)$$

Properties of Entropy and Mutual Information

- Entropy
  - $H(X_i)$ is maximized if all actions have equal utility
  - $H(X_i) = 0$ if all utility mass is concentrated on one action
- Mutual Information
  - $I(X_1, \ldots, X) \geq 0$
  - $I(X_1, \ldots, X) = 0$ if, and only if

$$w_{1:n}(a_{11}, \ldots a_{nn}) = \prod_{i=1}^{n} w_i(a_{ii})$$

which occurs if, and only if, all agents possess categorical utilities

For more details, see Cover and Thomas (1991)

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The Coordination Index

Definition

The *coordination index* is given by

$$\mathcal{C}(X_1, \ldots X_n) = \frac{I(X_1, \ldots, X_n)}{(n-1)H(X_1, \cdots, X_n)}$$

- $0 \leq \mathcal{C}(X_1, \ldots X_n) \leq 1$
- $\mathcal{C}(X_1, \ldots X_n) = 0$ if, and only if, all agents possess categorical utilities, even if $u_1 \equiv \cdots \equiv u_n$
- $\mathcal{C}(X_1, \ldots X_n) = 1$ if, and only if, all agents are slaved together
- A measure of the degree of coordination that arises *as a direct result of social influence*
  - An upper bound on the intrinsic ability of a group to coordinate
  - The coordination index is a property of the network — independent of solution concept

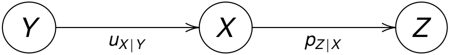
Outline

The Probability Syntax

- Conditional game theory uses the probability syntax in several non-traditional ways
  - Bayesian epistemology: conditionalization
  - Bayesian networks: utility networks
  - Dutch Book theorem (coherence): avoiding subjugation
  - Markov convergence theorem: cyclic networks
  - Shannon information theory: coordination index
- Conditional game theory may also be extended to account for stochastic agents
  - Stochastic agents may be directly embedded into a deterministic network
  - The probability syntax is used to interrelate both types of agents

Deterministic—Stochastic Interrelationships

- Consider the network



  - Y is a discrete random variable with probability mass function p_Y
  - X is a deterministic agent with conditional utility mass function $u_{X|Y}$
  - Z is a discrete random variable with conditional probability mass function $p_{Z|X}$
- Extending conditionalization
  - Conditioning intentions of deterministic agents on stochastic phenomena
  - Conditioning probabilities of stochastic phenomena on deterministic intentions

Stochastic Networks

- Let $\{X_1, \ldots, X_n\}$ denote a network of n deterministic agents with utilities defined over $\boldsymbol{\mathcal{A}}$
- Let $\{Y_1, \ldots, Y_m\}$ denote a set of discrete stochastic agents (random variables) where Y_j is defined over a finite set $\mathcal{B}_j$, $j = 1, \ldots, m$
- The set $\{X_1, \ldots, X_n, Y_1, \ldots, Y_m\}$ defined over $\mathcal{A}^n \times \mathcal{B}_1 \times \cdots \times \mathcal{B}_m$ is a *stochastic network*
- A *stochastic conjecture profile* is an array $(\mathbf{a}_1, \ldots, \mathbf{a}_n, b_1, \ldots, b_m) \in \mathcal{A}^n \times \mathcal{B}_1 \times \cdots \times \mathcal{B}_m$ such that $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a conjecture profile for $\{X_1, \ldots, X_n\}$ and $(b_1, \ldots, b_m)$ is a *stochastic profile* for $\{Y_1, \ldots, Y_m\}$
- A *stochastic coordination function* is a function $u_{1:n\,1:m}\colon \boldsymbol{\mathcal{A}}^n \times \mathcal{B}_1 \times \cdots \times \mathcal{B}_m \to [0, 1]$

Stochastic Coordination Function

- Define the parent sets

$$\mathrm{pa}(X_i) = \{X_{i_1}, \ldots, X_{i_{p_i}}, Y_{k_1}, \ldots, Y_{k_{q_i}}\}$$
$$\mathrm{pa}(Y_j) = \{X_{j_1}, \ldots, X_{j_{r_j}}, Y_{l_1}, \ldots, Y_{l_{s_j}}\}$$

- Define the conditioning conjectures

$$\boldsymbol{\alpha}_{\mathrm{pa}(i)} = (a_{i_1}, \ldots, a_{i_{p_i}}, b_{k_1}, \ldots, b_{k_{q_i}})$$
$$\mathbf{b}_{\mathrm{pa}(j)} = (a_{j_1}, \ldots, a_{j_{r_j}}, b_{l_1}, \ldots, b_{l_{s_j}})$$

- The *stochastic coordination function*

$$u_{1:n\,1:m}(\mathbf{a}_1, \ldots, \mathbf{a}_n, b_1, \ldots, b_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} u_{i|\mathrm{pa}(i)}(\mathbf{a}_i|\boldsymbol{\alpha}_{\mathrm{pa}(i)}) p_{j|\,\mathrm{pa}\,(j)}(b_j|\mathbf{b}_{\mathrm{pa}\,(j)})\,.$$

Expected Coordination Utility

- The *expected coordination function* is

$$\hat{u}_{1:n}(a_1, \dots, a_n) = \sum_{b_1, \dots, b_m} u_{1:n\,1:m}(a_1, \dots, a_n, b_1, \dots, b_m)\,.$$

- the *expected coordination utility* is

$$\hat{w}_{1:n}(a_{11}, \dots, a_{nn}) = \sum_{\sim a_{11}} \cdots \sum_{\sim a_{nn}} \hat{u}_{1:n}[(a_{11}, \dots, a_{1n}), \dots, (a_{n1} \dots, a_{nn})]$$

- The *expected individual decision function* is

$$\hat{w}_i(a_{ii}) = \sum_{\sim a_{ii}} \hat{w}_{1:n}(a_{11}, \dots, a_{1n})$$

Outline

Battle of the Sexes: Classical Formulation

A man (M) and a woman (W) plan to meet at one of two venues: the dog race (D) or the ballet (B)

		W	
		D	B
M	D	(4, 3)	(2, 2)
	B	(1, 1)	(3, 4)

4 = best; 3 = next-best; 2 = next-worst; 1 = worst

The best classical game theory can offer is a mixed strategy equilibrium

The Problem with the Classical Formulation

- No social context is expressed in the payoffs
- But if they are even contemplating being together, there must be some notion of social influence
- Suppose these individuals live in a society where the man defers to the woman regarding which event to attend
- In which case, the outcome (B, B) would be a unique focal point
- But if social relationships are relevant, then why not embed them into the preference model *ex ante*, rather than exogenously imposing them *ex post*?

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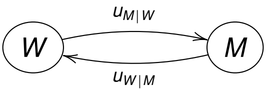
Battle of the Sexes as a Dissociated Cyclic Conditional Game

- Consider the cyclic model



- Conditional preferences for W and M

$$u_{W|M}(D|D) = 1-\alpha \quad u_{W|M}(B|D) = \alpha \qquad u_{M|W}(D|D) = 1 \quad u_{M|W}(B|D) = 0$$
$$u_{W|M}(D|B) = 0 \quad u_{W|M}(B|B) = 1 \qquad u_{M|W}(D|B) = \beta \quad u_{M|W}(B|B) = 1-\beta$$

- Transition matrices

$$T_{W|M} = \begin{bmatrix} 1-\alpha & 0 \\ \alpha & 1 \end{bmatrix} \qquad T_{M|W} = \begin{bmatrix} 1 & \beta \\ 0 & 1-\beta \end{bmatrix}$$

Cyclic Battle of the Sexes

- Closed-loop transition matrices

$$T_W = T_{W|M} T_{M|W} = \begin{bmatrix} 1-\alpha & \beta-\alpha\beta \\ \alpha & 1-\beta+\alpha\beta \end{bmatrix}$$

$$T_M = T_{M|W} T_{W|M} = \begin{bmatrix} 1-\alpha+\alpha\beta & \beta \\ \alpha-\alpha\beta & 1-\beta \end{bmatrix}$$

- The steady-state marginal utilities comprise the eigenvalues of the unique unity eigenvector

$$\overline{\mathbf{w}}_M = \begin{bmatrix} \overline{w}_M(D) \\ \overline{w}_M(B) \end{bmatrix} = \frac{1}{\alpha+\beta-\alpha\beta} \begin{bmatrix} \beta \\ \alpha-\alpha\beta \end{bmatrix}$$

$$\overline{\mathbf{w}}_W = \begin{bmatrix} \overline{w}_W(D) \\ \overline{w}_W(B) \end{bmatrix} = \frac{1}{\alpha+\beta-\alpha\beta} \begin{bmatrix} \beta-\alpha\beta \\ \alpha \end{bmatrix}$$

Coordinated Solution for Cyclic Battle of the Sexes

- The steady-state coordination utility is

$$\overline{w}_{MW}(a_M, a_W) = u_{M|W}(a_M|a_W)\overline{w}_W(a_W)$$

yielding (after dropping the normalizing factor $\frac{1}{\alpha+\beta-\alpha\beta}$)

$$w_{MW}(D,D) = \beta - \alpha\beta \quad w_{MW}(B,D) = 0$$
$$w_{MW}(D,B) = \alpha\beta \quad w_{MW}(B,B) = \alpha - \alpha\beta$$

- The individual decision function

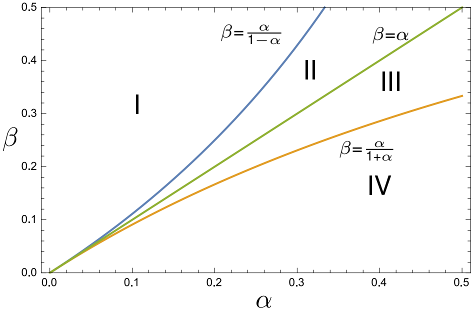
$$\overline{w}_M(a_M) = \sum_{a_W} \overline{w}_{MW}(a_M, a_W)$$
$$\overline{w}_W(a_W) = \sum_{a_M} \overline{w}_{MW}(a_M, a_W)$$

yielding

$$\overline{w}_M(D) = \beta \qquad u_M(B) = \alpha - \alpha\beta$$
$$\overline{w}_W(D) = \beta - \alpha\beta \quad u_M(B) = \alpha$$

Coordinated Solution for Cyclic Battle of the Sexes

(α, β) Region	Coordination utility ordering	Individual M ordering	Individual W ordering	*GTR* decision	Consensus choice?
I	$D, D \succ_{MW} B, B$	$D \succ_M B$	$D \succ_W B$	D	yes
II	$D, D \succ_{MW} B, B$	$B \succ_M D$	$D \succ_W B$	D	no
III	$B, B \succ_{MW} D, B$	$B \succ_M D$	$D \succ_W B$	B	no
IV	$B, B \succ_{MW} D, B$	$B \succ_M D$	$B \succ_W D$	B	yes



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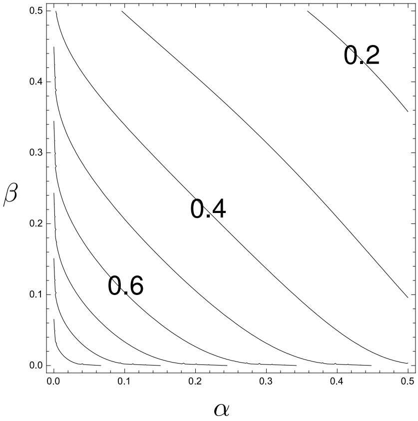
Coordination Index for Cyclic Battle of the Sexes



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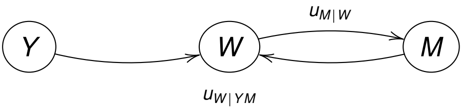
Stochastic Battle of the Sexes

Let Y be a discrete random variable defined over $\{A, S\}$, for ballet tickets available with $p_Y(A) = \gamma$ or sold out with $p_Y(S) = 1 - \gamma$



- Invoking invariance and dissociaton, the coordination function is

$$u_{MWY}(a_M, a_W, y) = u_{MY|W}(a_M, y|a_W)u_W(a_W)$$
$$u_{MWY}(a_M, a_W, y) = u_{W|MY}(a_W|a_M, y)u_{MY}(a_M, y)$$

- Computing the marginals, we obtain

$$u_W(a_W = \sum_{a_M, y} u_{MY|W}(a_M, y|a_W)u_W(a_W)$$
$$u_{MY}(a_M, y) = \sum_{a_W} u_{W|MY}(a_W|a_M, y)u_{MY}(a_M, y)$$

or, in matrix form

$$\mathbf{u}_W = T_{W|MY}\mathbf{u}_{MY}$$
$$\mathbf{u}_{MY} = T_{MY|W}\mathbf{u}_W$$

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Outline

Prisoner's Dilemma: Classical Model

- Joint action set $\mathcal{A} = \{(C, C), (C, D), (D, C)), (D, D)\}$
- Axelrod (1984) payoff matrix
  - R = reward for mutual cooperation
  - S = sucker's payoff
  - T = temptation to defect
  - P = punishment for mutual defection

		X_c	
		C	D
X_r	C	R, R	S, T
	D	T, S	P, P

$T > R > P > S$ and $R > (T + S)/2$

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Prisoner's Dilemma: Conditional Game Model

- Assumption #1: If X_i conditionally conjectures (D, D) then X_j will conditionally conjecture (D, D), $i, j, \in \{r, c\}$
- Assumption #2: If X_i conditionally conjectures (C, C) then X_j will conditionally conjecture (C, C), $i, j, \in \{r, c\}$
- Assumption #3: If X_i conditionally conjectures (D, C) or (C, D), then X_j will conditionally conjecture according to the classical Prisoner's Dilemma preference ordering

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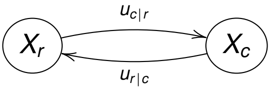
Prisoner's Dilemma: Conditional Game Model

- Model structure



- Conditional utilities ordering structure
  Key: 4 = best, 3 = next-best, 2 = next-worst, 1 = worst

$u_{r|c}(a_{rr}, a_{rc}|a_{cr}, a_{cc})$

	a_{cr}, a_{cc}			
a_{rr}, a_{rc}	C, C	C, D	D, C	D, D
C, C	4	3	3	2
C, D	1	1	1	1
D, C	3	4	4	3
D, D	2	2	2	4

(a)

$u_{c|r}(a_{cr}, a_{cc}|a_{rr}, a_{rc})$

	a_{rr}, a_{rc}			
a_{cr}, a_{cc}	C, C	C, D	D, C	D, D
C, C	4	3	3	2
C, D	3	4	4	3
D, C	1	1	1	1
D, D	2	2	2	4

(b)

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Prisoner's Dilemma: Conditional Utilities

- parameterize the conditional payoff matrices: $1/2 < \alpha < \pi$

$$\pi \sim T$$
$$\alpha \sim R$$
$$1-\alpha \sim P$$
$$1-\pi \sim S$$

		X_c	
		C	D
X_r	C	$\alpha, , \alpha$	$1-\pi, \pi$
	D	$\pi, 1-\pi$	$1-\alpha, 1-\alpha$

$\pi > \alpha > 1-\alpha > 1-\pi$

- State-to-state transition matrices

$$T_{r|c} = 1/2 \begin{bmatrix} \pi & \alpha & \alpha & 1-\alpha \\ 1-\pi & 1-\pi & 1-\pi & 1-\pi \\ \alpha & \pi & \pi & \alpha \\ 1-\alpha & 1-\alpha & 1-\alpha & \pi \end{bmatrix}$$

$$T_{c|r} = 1/2 \begin{bmatrix} \pi & \alpha & \alpha & 1-\alpha \\ \alpha & \pi & \pi & \alpha \\ 1-\pi & 1-\pi & 1-\pi & 1-\pi \\ 1-\alpha & 1-\alpha & 1-\alpha & \pi \end{bmatrix}$$

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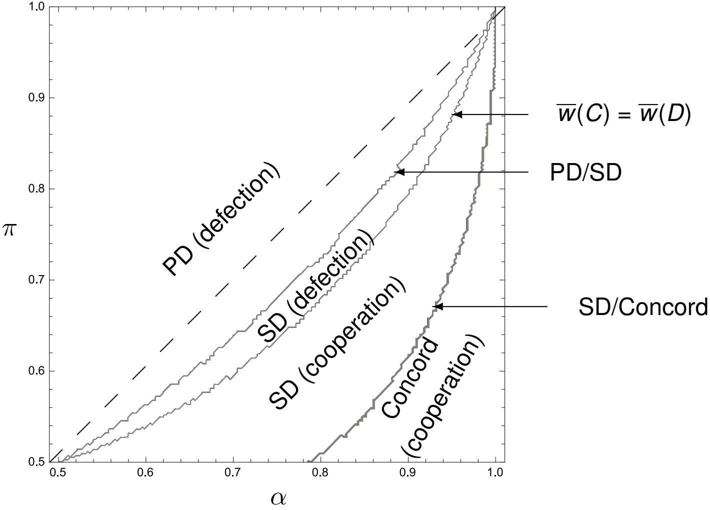
Prisoner's Dilemma: Results



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Prisoner's Dilemma Coordinatability Index

- $\mathcal{C}(X_r, X_c) \approx 0$ for all $(\alpha, \pi) \in (1/2, 1) \times (1/2, 1)$
- Bacharach's hypothesis:

The overwhelmingly most frequent example of a scenario in which a sense of interdependence is said to promote group identification is certainly a case of strong interdependence. It is the Prisoner's Dilemma.

— M. Bacharach
Beyond Individual Choice

- But this analysis suggests that the Prisoner's Dilemma is a model of group behavior that is virtually impervious to team formation: the opportunity to exploit clashes with the opportunity to cooperate — coordination is problematic

Outline

New Applications of Existing Mathematics

The manner in which mathematical theories are applied does not depend on preconceived ideas; it is purposeful technique depending on, and changing with, experience.

— William Feller

An Introduction to Probability Theory and Its Applications

- Conditionalization logic applied to utility theory
- Bayesian network theory applied to agent networks
- The Dutch Book theorem applied to coherent coordination
- Markov convergence theory applied to reciprocity
- Shannon information theory applied to coordinatability
- Seamlessly combining the stochastic and deterministic aspects of a game with a common syntax

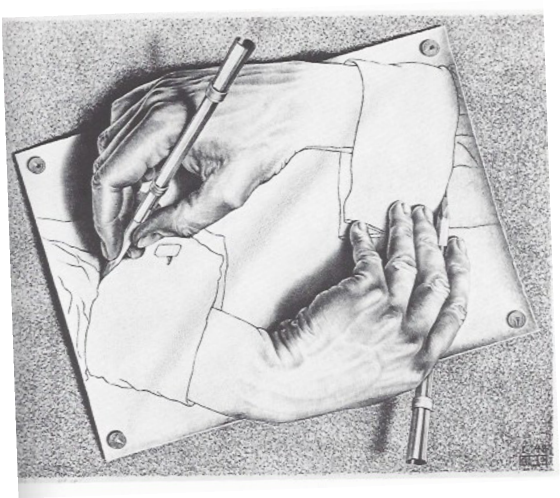
Contrasts Between Classical and Conditional Games

Issue	Classical Game Theory	Conditional Game Theory
Game structure	Divison of labor: preference model and solution concept defined separately	Social influence integrated into conditional preference model
Preference model	Categorical — fixed and immutable	Conditional — modeled after Bayesian conditionalization
Rationality concept	Individual rationality with regard to material performance	Individual rationality with regard to material benefit as modulated by social influence
Solution concept	Individual: constrained optimization (Nash equilibrium) Group: individuals possess a concept of shared intentions (team reasoning)	Individual: maximize the coordinated individual rationality utility Group: maximize coordination utility (generalized team reasoning)
Coordination concept	Extrinsic: exogenously defined social solution concept applied to payoffs	Intrinsic: endogenously emerges as social influence propagates through the network

State of the Art of Conditional Game Theory

- A framework for additional ways to model behavior
  - *Ex ante* incorporation of social influence into preferences
  - Permits responsive rather than reactive behavior
- A way to formalize bargaining (virtual or direct)
- Toward an explication of coordination

Thank You



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Stochastic Battle of the Sexes

Let Y be a discrete random variable defined over $\{A, S\}$, for ballet tickets available with $p_Y(A) = \gamma$ or sold out with $p_Y(S) = 1 - \gamma$

- Invoking invariance and dissociaton, the coordination function is

$$u_{MWY}(a_M, a_W, y) = u_{MY|W}(a_M, y|a_W)u_W(a_W)$$
$$u_{MWY}(a_M, a_W, y) = u_{W|MY}(a_W|a_M, y)u_{MY}(a_M, y)$$

- Computing the marginals, we obtain

$$u_W(a_W = \sum_{a_M, y} u_{MY|W}(a_M, y|a_W)u_W(a_W)$$
$$u_{MY}(a_M, y) = \sum_{a_W} u_{W|MY}(a_W|a_M, y)u_{MY}(a_M, y)$$

which, in matrix notation, becomes

$$\mathbf{u}_W = T_{W|MY}\mathbf{u}_{MY}$$

Solving the Stochastic Battle of the Sexes

Closing the loop around W yields

$$\mathbf{u}_W = T_{W|MY}\mathbf{u}_{MY} = T_{W|MY}T_{MY|W}\mathbf{u}_W$$

W's preferences are conditioned on Y and M yielding

$$\begin{aligned}
u_{W|YM}(D|A,D) &= 1-\alpha\\
u_{W|YM}(B|A,D) &= \alpha\\
u_{W|YM}(D|A,B) &= 0\\
u_{W|YM}(B|A,B) &= 1\\
u_{W|YM}(D|S,D) &= 1\\
u_{W|YM}(B|S,D) &= 0\\
u_{W|YM}(D|S,B) &= 1\\
u_{W|YM}(B|S,B) &= 0
\end{aligned}
\quad \text{or} \quad
T_{W|YM} = \begin{bmatrix} 1-\alpha & 0 & 1 & 1 \\ \alpha & 1 & 0 & 0 \end{bmatrix}$$

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Solving the Stochastic Battle of the Sexes

Assume that Y and M are conditionally independent, given W,

$$u_{YM|W}(y, a_M|a_W) = p_{Y|W}(y|a_W)u_{M|W}(a_M|a_W) = p_Y(y)u_{M|W}(a_M|a_W)$$

yielding

$$\begin{aligned}
u_{YM|W}(A, D|D) &= \gamma \\
u_{YM|W}(A, B|D) &= 0 \\
u_{YM|W}(S, D|D) &= 1 - \gamma \\
u_{YM|W}(S, B|D) &= 0 \\
u_{YM|W}(A, D|B) &= \gamma\beta \\
u_{YM|W}(A, B|B) &= \gamma(1 - \beta) \\
u_{YM|W}(S, D|B) &= (1 - \gamma)\beta \\
u_{YM|W}(S, B|B) &= (1 - \gamma)(1 - \beta)
\end{aligned}$$

or

$$T_{YM|W} = \begin{bmatrix} \gamma & \gamma\beta) \\ 0 & \gamma(1 - \beta) \\ 1 - \gamma & (1 - \gamma)\beta \\ 0 & (1 - \gamma)(1 - \beta) \end{bmatrix}$$

Solving the Stochastic Battle of the Sexes

- Closed-loop transition matrix

$$T_W = T_{W|YM} T_{YM|W} = \begin{bmatrix} 1 - \alpha\gamma & 1 - \gamma + \beta\gamma - \alpha\beta\gamma \\ \alpha\gamma & \gamma - \beta\gamma + \alpha\beta\gamma \end{bmatrix}$$

- Expected steady-state decision functions

$$\begin{bmatrix} \overline{w}_W(D) \\ \overline{w}_W(B) \end{bmatrix} = \begin{bmatrix} \frac{1-\gamma+\beta\gamma-\alpha\beta\gamma}{1-\gamma+\beta\gamma+\alpha\gamma-\alpha\beta\gamma} \\ \frac{\alpha\gamma}{1-\gamma+\beta\gamma+\alpha\gamma-\alpha\beta\gamma} \end{bmatrix}$$

$$\begin{bmatrix} \overline{w}_M(D) \\ \overline{w}_M(B) \end{bmatrix} = \begin{bmatrix} \frac{1-\gamma-\beta\gamma}{1-\gamma+\beta\gamma+\alpha\gamma-\alpha\beta\gamma} \\ \frac{\alpha\gamma-\alpha\beta\gamma}{1-\gamma+\beta\gamma+\alpha\gamma-\alpha\beta\gamma} \end{bmatrix}$$