Portfolio Constraints, Differences in Beliefs and Bubbles *

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Abstract

I propose an arbitrage-based theory of rational bubbles in economies with general portfolio constraints and differences in beliefs. Trading restrictions and speculation due to asymmetric information and heterogeneous beliefs do not cause bubbles. Low interest rates are again needed for bubbles to exist, as in economies with symmetric information and agents subject to borrowing constraints (Santos and Woodford 1997).

Keywords: rational bubbles, speculative bubbles, asymmetric information, heterogeneous beliefs, high interest rates, resale option

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1 Introduction

Rapid increases in asset prices, followed by collapses, are often seen as evidence of bubbles. A bubble measures the portion of an asset price in excess of its fundamental value, calculated as the expected discounted present value of its dividends.

Santos and Woodford (1997) showed that in standard stochastic dynamic general equilibrium models with *symmetric information* and agents subject to *borrowing constraints*,[1] bubbles can be ruled out on assets in positive supply when the interest rates are *high*, making the present value of aggregate consumption finite.[2] The finite present value of aggregate endowment (the high interest rates assumption) is tantamount in their environment to the existence of a sufficiently productive asset (or, more generally, of a portfolio with positive holdings of the assets) with dividend (payoff) in excess of a fixed fraction of the aggregate endowment.

The outline of their argument is that bubbles grow on average at the rate of interest rates. With high interest rates, the bubble must become very large relative to aggregate endowment, even if this happens with small probability. But this is incompatible with the presence of optimizing, forward looking agents, who do not allow their financial wealth to become too large relative to the present value of their future consumption.

I show that this reasoning ruling out bubbles is very robust. It can be extended to environments with general portfolio constraints, asymmetric information and heterogeneous beliefs. This seems surprising, since allowing for more severe portfolio restrictions than the borrowing constrains considered by Santos and Woodford (1997) might improve the chances for bubbles, as it could be harder to short the overvalued assets. Similarly, one expects that the presence of heterogeneous beliefs leads to speculation, and hence to overvalued assets and bubbles. Finally, asymmetrically informed agents might trade in overvalued assets as more informed agents expect to

[1] Borrowing constraints impose lower (negative) bounds on an agent's end of period financial wealth.

[2] The results of Santos and Woodford (1997) were anticipated by Scheinkman (1977) and Kocherlakota (1992), and later refined by Huang and Werner (2000) for deterministic economies. Montrucchio and Privileggi (2001) also show that under mild assumptions on agent's preferences, bubbles cannot exist in a representative agent economy.

sell them before the crash to less informed agents (the "greater fool" theory).

Santos and Woodford (1997, Theorem 3.1) prove that there exists a stochastic discount factor (*SDF* henceforth) compatible with the absence of arbitrage opportunities such that the discounted present value of an asset's dividends (using this SDF) equals its price. Moreover, if the agents are sufficiently impatient,[3] in the sense that they are always willing to trade a fixed fraction of all future consumption in exchange for the current aggregate endowment, then the price of an asset in positive supply is always equal to its fundamental value, for every SDF compatible with the absence of arbitrage (Santos and Woodford 1997, Theorem 3.3).

Portfolio constraints are needed for the existence of bubbles, otherwise agents would short the overpriced assets. Therefore bubbles can only be defined by taking into account the underlying trading restrictions. With general portfolio constraints, two main difficulties arise. First, arbitrage opportunities can exist in equilibrium (see Example 4.4.1 with short sale constraints in Leroy and Werner 2001). Second, there might not exist an SDF that makes the price of the assets equal to the sum of expected discounted value of next period dividend and resale price. In other words, with more general portfolio constraints (rather than borrowing constraints), the fundamental theorem of asset pricing may not hold (this is the case, for example, for short sale constraints).

Despite these difficulties, I show that an arbitrage-based theory of bubbles still exists, and it reduces to the approach of Santos and Woodford (1997) for the case of borrowing constraints. Indeed, *unrestricted* arbitrage opportunities, that is arbitrage opportunities that can be added to any feasible trading strategy and scaled up arbitrarily cannot exist. Equivalently, there cannot be arbitrage opportunities in the *recession* cones of agents' constraints. A Farkas-Stiemke lemma for cones establishes the existence of agent specific SDFs that can be used in discounting dividends. Due to heterogeneity in agents' constraints and information, the notion of *high* interest rates is now agent specific. Interest rates are high from the point of view of an agent

[3]The impatience assumption was introduced by Magill and Quinzii (1994) and Levine and Zame (1996) and used to prove the existence of equilibrium in economies with infinite horizon and incomplete markets.

if the present value of aggregate endowment is finite under all of his SDFs. Paralleling Santos and Woodford (1997), high interest rates for an agent amount to the existence of a portfolio in the recession cone of his constraints with payoff in excess of the aggregate endowment. This follows from the duality result in Huang (2002).

Given an arbitrary SDF derived from the absence of unrestricted arbitrages for a given agent, the price of an asset can be decomposed into three nonnegative components. The first term is the discounted present value of the asset's dividends. When the SDF is given by the intertemporal marginal rates of substitution of the agent, it represents what the agent would be willing to pay if he was forced to maintain his holdings of one unit of the asset forever.

The second term in the decomposition represents the *resale option* afforded to the agent by being long one unit of the asset. The term was coined by Scheinkman and Xiong (2003) and captures the excess over what the agent is willing to pay if he cannot trade the asset in the future. Pascoa, Petrassi, and Torres-Martinez (2011) and Araujo, Páscoa, and Torres-Martínez (2011) refer to it as the *shadow price* of agent's constraints, as it measures deviations from the fundamental theorem of asset pricing. It represents the value of all future *services* in relaxing (binding) financial constraints. Cochrane (2002) further interprets the resale option as the *convenience yield* generated by being long one unit of the asset, as holding inventories helps to better smooth demand in the presence of shorting restrictions.

Finally, the third component is given by the asymptotic expected discounted value of the asset, and will be referred to as a bubble (under the chosen SDF), whenever it is nonzero. In discounted terms, the bubble is a martingale. This is the usual definition of a rational bubble found in the literature, and coincides with the definition of Santos and Woodford (1997), since with borrowing constraints the resale option is always zero.

The first non-existence of bubbles result (Theorem 3.1) shows that there are no bubbles in assets in positive supply from the point of view of uninformed (having only public information) agents if they perceive interest rates as high. Theorem 3.1 of Santos and Woodford (1997) obtains as a particular case if agents have symmetric information and homogeneous beliefs, and face borrowing constraints. Two corollaries follow for particular portfolio constraints, without requiring the presence of uninformed agents, but at the cost of additional assumptions. If agents face no *short sales* restrictions and if there is an agent with high interest rates that is unconstrained in a given asset infinitely often, then there exists a SDF associated to that agent under which the asset is bubble-free. Alternatively, if agents face *debt* or *borrowing* constraints and markets are *complete* from the point of view of a (hypothetical) uninformed agent having high interest rates, then there are no bubbles in assets in positive supply under *any* agent specific SDF.

By imposing the same form of impatience on agents as Santos and Woodford (1997), the previous non-existence results can be substantially strengthened. The presence of uninformed agents is not needed. Moreover, bubbles are absent under *any* SDF associated to an agent with high interest rates (Theorem 3.4). This result extends Theorem 3.3 in Santos and Woodford (1997) to economies with differences in beliefs and general portfolio constraints.

The absence of bubbles under asymmetric information was anticipated by Tirole (1982), in a model with risk neutral agents and only one asset. However, as pointed out by Kocherlakota (1992), he overlooked the crucial need for portfolio restrictions. Without them, agents can run Ponzi schemes and no equilibrium (and of course, no bubbles) can exist. Yu (1998) allowed for asymmetric information in the framework of Santos and Woodford (1997) with agents subject to borrowing constraints and showed that their Theorem 3.3, on the non-existence of bubbles when agents are impatient, is still true. By contrast, the non-existence of bubbles results of this paper applies to economies with a variety of portfolio constraints (including borrowing, debt, short sale or margin constraints) and heterogeneous beliefs, in addition to asymmetric information. Moreover, some of the results in-here do not make use of the impatience assumption.

In apparent contradiction to my message, Harrison and Kreps (1978) and a large body of subsequent literature reviewed in Xiong (2013) (see, for example Morris 1996, Scheinkman and Xiong 2003) argue that "speculative" bubbles exist in economies with short sale constraints and heterogeneous beliefs. In the language of this paper, speculative bubbles refer to resale options (convenience yields). Thus an asset has a speculative bubble if, for each agent, his resale option (convenience yield) calculated using his own intertemporal marginal rates of substitution (*IMRS* henceforth) as SDF is positive.

There is nothing special about *heterogeneous* beliefs or *short sale* constraints in generating "speculative bubbles". For example, in limited enforcement economies with *homogeneous* beliefs and (endogenous) *debt* constraints (Alvarez and Jermann 2000), at each period of time the (unique) stochastic discount factor is given by the highest IMRS of all agents (or equivalently, the IMRS of the unconstrained agents). The price of an asset is equal to the present value of dividends when discounted using this "natural" (market) stochastic discount factor, therefore there are no resale options and (rational) bubbles under this SDF. However the price of the asset strictly exceeds its present value of dividends discounted using the IMRS of any given agent, and therefore "speculative bubbles" exist. In fact most of the models with (binding) financial frictions encountered in the literature display "speculative bubbles". This suggests that indeed, resale option or convenience yield is a better term (rather than "bubble"). This paper reserves the term bubble for "rational" (asymptotic) bubbles, which are discounted martingales, and can occur only in economies with infinitely many trading dates.

Under the arbitrage-based valuation theory developed here, in both types of economies mentioned before (heterogeneous beliefs and short sale constraints, or limited enforcement economies with homogeneous beliefs and debt limits), the set of valid SDFs contains also the "mixture" of agents' IMRS. In particular, at each period, the IMRS of an unconstrained agent who chooses to hold the asset can be used for discounting next period's dividend. Therefore there are arbitrage-free SDFs that martingale-price the assets in positive supply. Under such SDFs there are no resale options, and hence there are no speculative bubbles (Proposition 3.5). Moreover, there are also no rational bubbles under such SDFs, by Theorem 3.4, if interest rates are high (from the point of view of any agent). Thus with symmetric information, the price of an asset in positive supply cannot be unambiguously higher than the discounted present value of its dividends, even with heterogeneous beliefs and short sale constraints.

However, with asymmetric information and short sale constraints, resale options can be positive under any arbitrage-free SDF (including mixtures of agents' IMRS). Such examples with positive resale options are constructed in Allen, Morris, and Postlewaite (1993). The reason is that the "unconstrained" agent holding positive amounts of the asset pools the information of several agents. With asymmetric information, only fully informed agents can, in general, rationalize the price of the asset as being equal to the present value of its dividends.

The results of this paper indicate that asymmetric information, heterogeneous beliefs or short sale constraints do not cause rational bubbles. The need for low interest rates in order to sustain bubbles remains. Recent work suggests that low interest rates might be more prevalent than thought before. On the theoretical front, we now know that environments with limited enforcement lead naturally to low interest rates and robust bubbles (Hellwig and Lorenzoni 2009, Bidian 2011, Bidian 2014). On the empirical front, Geerolf (2013), by using a new data set, overturned the finding of Abel, Mankiw, Summers, and Zeckhauser (1989) that capital in developed economies is sufficiently productive. The return on capital in excess of investment as a fraction of GDP is not in general positive and bounded away from zero, and therefore bubbles cannot be ruled out.

2 Arbitrage-based definition of bubbles

2.1 Setup

The economy has an infinite horizon and is stochastic. The time periods are indexed by the set $\mathbb{N} := \{0, 1, \ldots\}$. There is a single consumption good and a finite number I of consumers that have asymmetric information and heterogeneous beliefs. Agent i's information is described by a filtration $(\mathcal{F}_t^i)_{t=0}^{\infty}$, which is an increasing sequence of σ-algebras on the set of states of the world Ω. Each σ-algebra $\mathcal{F}_t^i$ is interpreted as agent's i information at period t and is finite. Thus for $\omega \in \Omega$ and $t \in \mathbb{N}$, the set of states that i believes are possible at t if the true state is ω is $\mathcal{F}_t^i(\omega) := \cap\{A \in$

$\mathcal{F}_t^i \mid \omega \in A\}$.[4] The beliefs of i are given by a probability P^i on $(\Omega, \sigma(\cup_{t=0}^{\infty}\mathcal{F}_t^i))$, with $P^i(\mathcal{F}_t^i(\omega)) > 0$, for all t and ω. The heterogeneity in beliefs can be caused by different priors or by distortions in updating during the learning process.

The *join (pool) filtration* $\mathcal{F} = (\mathcal{F}_t)_{t=0}^{\infty}$ is defined as the aggregate information of all agents: $\mathcal{F}_t := \sigma(\cup_{i\in I}\mathcal{F}_t^i)$, for all t. Let $\mathcal{F}_\infty := \sigma(\cup_{t=0}^{\infty}\mathcal{F}_t)$, and let P be a probability on $\mathcal{F}_\infty$ such that $P(\mathcal{F}_t^i(\omega)) > 0$ for all i, t, ω.[5] The *meet filtration* $(\mathcal{F}_t^m)$ captures the *public information*, or the *common knowledge* of all agents: $\mathcal{F}_t^m := \cap_i \mathcal{F}_t^i$, for all t.

Let $\mathcal{G} = (\mathcal{G}_t)_{t=0}^{\infty}$ be a filtration on $(\Omega, \mathcal{F}_\infty)$. A sequence $x = (x_t)_{t\in\mathbb{N}}$ of random variables ($\mathcal{F}_\infty$-measurable real-valued functions) is a *stochastic process adapted to* $(\mathcal{G}_t)_{t=0}^{\infty}$ if for each $t \in \mathbb{N}$, x_t is $\mathcal{G}_t$-measurable. $X(\mathcal{G})$ is the set of all stochastic processes adapted to $\mathcal{G}$, and $X_+(\mathcal{G})$ (respectively $X_{++}(\mathcal{G})$) contains the processes $x \in X(\mathcal{G})$ such that $x_t \geq 0$ P-almost surely (respectively $x_t > 0$ P-almost surely) for all $t \in \mathbb{N}$. All statements, equalities, and inequalities involving random variables are assumed to hold only P-almost surely, and this qualifier is omitted.

Agent i has endowments $e^i \in X_+(\mathcal{F}^i)$, and his continuation utility at t provided by a consumption stream $c \in X_+(\mathcal{F}^i)$ is $U_t^i(c) := E_{\mathcal{F}_t^i}^{P^i} \sum_{s\geq t} \bar{u}_s^i(c_s)$ where $\bar{u}_s^i : \mathbb{R}_+ \to \mathbb{R}$ is continuous, increasing and concave for all $s \in \mathbb{N}$ and $E_{\mathcal{F}_t^i}^{P^i}(\cdot)$ is the conditional expectation with respect to probability P^i and σ-algebra $\mathcal{F}_t^i$. Let $\beta_t^i > 0$ be the Radon-Nikodym derivative of P^i with respect to P, when restricted to $\mathcal{F}_t^i$, and set $u_t^i := \beta_t^i \bar{u}_t^i$. It follows that $U_t^i(c) = E_{\mathcal{F}_t^i} \sum_{s\geq t} u_t^i(c_t)$ where $E_{\mathcal{F}_t^i}(\cdot)$ is to expectation with respect to P, conditional on $\mathcal{F}_t^i$. Without loss of generality, throughout the paper the expectations with respect to agents' beliefs are converted to expectations with respect to P, via the use of Radon-Nikodym derivatives as above.

There is a finite number J of infinitely lived, disposable securities, traded at every date. The dividends of asset $j \in \{1, 2, \ldots, J\}$ are described by the common knowledge process $d^j \in X_+(\mathcal{F}^m)$. The ex-dividend price per share of firm j is a process $p^j \in X_+(\mathcal{F}^m)$. Thus it is assumed that agents already extracted all the available information from prices and dividends, refining their information accordingly. Start-

[4] Using the usual "event tree" terminology, $\mathcal{F}_t^i(\omega)$ is the date t node containing state ("leaf") ω.

[5] For example, each P^i can be defined on $\mathcal{F}_\infty$ from the start and P can be taken to be the average belief.

ing with "post-extraction" prices and dividends is without loss of generality, and simplifies the notation. Denote by $d = (d^1, \dots, d^J) \in X_+^J(\mathcal{F}^m)$ the dividend vector process, and by $p = (p^1, \dots, p^J) \in X_+^J(\mathcal{F}^m)$ the price vector process.

Consumer i has an initial endowment $\theta_{-1}^i = (\theta_{-1}^{i,1}, \dots, \theta_{-1}^{i,J})'$ of securities ($\mathcal{F}_0^i$-measurable random J-dimensional random vector) and his trading strategy is represented by a vector process $\theta^i = (\theta^{i,1}, \dots, \theta^{i,J})' \in X^J(\mathcal{F}^i)$. Securities are in nonnegative supply, thus $\bar{\theta} := \sum_{i=1}^I \theta_{-1}^i \in \mathbb{R}_+^J$. Let $e := \sum_i e^i$ be the aggregate endowment and $\tilde{e} := e + d\bar{\theta}$ be the aggregate goods supply. The aggregate endowment is assumed to be common knowledge, that is $e \in X_+(\mathcal{F}^m)$. Consumer i's portfolios at t are restricted to a set $\Theta_t(p; \mathcal{F}^i)$ of $\mathcal{F}_t$-measurable and J-dimensional random vectors. These portfolio constraints are meant to prevent Ponzi schemes. It follows that agent's trading strategies are restricted to the set $\Theta(p; \mathcal{F}^i) := \prod_{t=0}^\infty \Theta_t(p; \mathcal{F}^i) \subset X^J(\mathcal{F}^i)$. I assume that $0 \in \Theta(p; \mathcal{F}^i)$, thus agents can always choose not to trade.

The budget constraint of agent i facing prices p is

$$B^i(p) = \left\{(c^i, \theta^i) \in X_+(\mathcal{F}^i) \times \Theta(p; \mathcal{F}^i) \mid c_t^i + p_t\theta_t^i \leq e_t^i + (p_t + d_t)\theta_{t-1}^i, \forall t \geq 0\right\}. \tag{2.1}$$

The definition of a *rational expectations equilibrium* is standard and coincides with the one used by Allen, Morris, and Postlewaite (1993) in their particular framework.

Definition 2.1. *A vector* $\left(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ *of prices and agents' consumption and trading strategies is an equilibrium if:*

i. The consumption and trading strategies of each agent i are feasible and optimal: $(c^i, \theta^i) \in B^i(p)$*, and* $U_0^i(c^i) \geq U_0^i(\hat{c}^i)$*, for all* $(\hat{c}^i, \hat{\theta}^i) \in B^i(p)$*.*

ii. Markets for goods and securities clear,

$$\sum_{i=1}^I c_t^i = \sum_{i=1}^I e_t^i + d_t \cdot \bar{\theta}, \quad \sum_{i=1}^I \theta_t^i = \bar{\theta}, \quad \forall t \in \mathbb{N}. \tag{2.2}$$

2.2 State price densities

A process $\theta \in X^J(\mathcal{F}^i)$ is an *arbitrage opportunity* for agent i at prices p if

$$(p_t + d_t)\theta_{t-1} - p_t\theta_t \geq 0, \forall t \geq 0,$$

with at least one of the inequalities being strict with positive probability, where $\theta_{-1} := 0 \in \mathbb{R}^J$.

Consider an equilibrium $\left(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$. With portfolio constraints, the strict monotonicity of an agent i's preferences does not guarantee the absence of *feasible* (belonging to $\Theta(p;\mathcal{F}^i)$) arbitrage opportunities (see, for example, Leroy and Werner 2001). However, it guarantees the absence of *unrestricted* arbitrage opportunities in $R(\Theta(p;\mathcal{F}^i))$, the *recession cone*[6] of $\Theta(p;\mathcal{F}^i)$:

$$R(\Theta(p;\mathcal{F}^i)) := \left\{\theta \in X^J(\mathcal{F}^i) \mid \Theta(p;\mathcal{F}^i) + \lambda\theta \subset \Theta(p;\mathcal{F}^i), \forall \lambda > 0\right\}.$$

Indeed, if $\theta \in R(\Theta(p;\mathcal{F}^i))$ is an arbitrage opportunity, agent i would alter his trading strategy to $\theta^i + \theta$, which is feasible and provides strictly higher utility.[7]

Throughout the paper, it is assumed that the constraints do no prevent agents from adding positive holdings of shares to their portfolios, and that the recession cones are polyhedral:[8]

Assumption 2.1. *For each $\omega \in \Omega$ and $t \in \mathbb{N}$, the set $\{\theta_t(\omega) \mid \theta \in R(\Theta(p;\mathcal{F}^i))\}$ ($\subset \mathbb{R}^J$) is a polyhedral cone containing $\mathbb{R}^J_+$.*

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[6]The recession cone of a subset D of a vector space X is the set $R(D) := \{d \in X \mid D + \lambda d \subset D, \forall \lambda > 0\}$.

[7]If agent's i utilities u_t^i go to infinity as consumption goes to infinity, it follows that *unlimited* arbitrage opportunities must be absent. An unlimited arbitrage opportunity is an arbitrage opportunity that belongs to the *cone of unlimited portfolios* in $\Theta(p;\mathcal{F}^i)$:

$$\bar{R}(\Theta(p;\mathcal{F}^i)) := \left\{\theta \in \Theta(p;\mathcal{F}^i) \mid \lambda\theta \subset \Theta(p;\mathcal{F}^i), \forall \lambda > 0\right\}.$$

All the results of this paper hold if recession cones $R(\Theta(p;\mathcal{F}^i))$ are substituted by $\bar{R}(\Theta(p;\mathcal{F}^i))$ (if the additional utility unboundedness assumption is made). For the canonical cases of borrowing, debt and short sales constraints, $R(\Theta(p;\mathcal{F}^i)) = \bar{R}(\Theta(p;\mathcal{F}^i))$, as it will be seen next.

[8]A polyhedral cone is the intersection of a finite number of half-spaces of a Euclidean space that is stable under addition and multiplication by nonnegative real numbers.

Absence of unrestricted arbitrages implies, in particular, that for each $t \in \mathbb{N}$, arbitrage opportunities in $R(\Theta(p;\mathcal{F}^i))$ which are equal to zero at all periods except t (henceforth referred to as *one-period* arbitrage opportunities) must be absent. A Stiemke lemma for cones (Appendix A) shows that the absence of one-period arbitrage opportunities in $R(\Theta(p;\mathcal{F}^i))$ is equivalent to the existence of an $a \in X_{++}(\mathcal{F}^i)$ such that

$$\left(p_t - E_{\mathcal{F}^i_t}\frac{a_{t+1}}{a_t}(p_{t+1}+d_{t+1})\right)\theta_t \geq 0, \forall t \in \mathbb{N}, \forall \theta \in R(\Theta(p;\mathcal{F}^i)). \tag{2.3}$$

Let $A(p;\mathcal{F}^i)$ be the set of all processes $a \in X_{++}(\mathcal{F}^i)$ satisfying equation (2.3), with $a_0 := 1$ by normalization. An element of $A(p;\mathcal{F}^i)$ is called a *state price density* given the information structure $\mathcal{F}^i$.

The *discounted present value* of a nonnegative process $x \in X_+(\mathcal{F})$ under a state price density $a \in A(p;\mathcal{F}^i)$ given the information structure $\mathcal{F}^i$ is the process $f(a,x;\mathcal{F}^i) \in X_+(\mathcal{F}^i)$ defined by

$$f_t(a,x;\mathcal{F}^i) := \frac{1}{a_t}E_{\mathcal{F}^i_t}\sum_{s>t} a_s x_s, \forall t \in \mathbb{N}. \tag{2.4}$$

Denote by $\pi_t(x;\mathcal{F}^i)$ the supremum across all state price densities in $A(p;\mathcal{F}^i)$ of the discounted present value of x at t:

$$\pi_t(x;\mathcal{F}^i) := \sup_{a\in A(p;\mathcal{F}^i)} f_t(a,x;\mathcal{F}^i). \tag{2.5}$$

Notice that for computing $\pi_t(x;\mathcal{F}^i)$, it is enough to take the supremum in (2.5) over the maximal elements of the set $A(p;\mathcal{F}^i)$, defined as

$$\bar{A}(p;\mathcal{F}^i) := \{a \in A(p;\mathcal{F}^i) \mid \nexists a' \in A(p;\mathcal{F}^i) \setminus \{a\} \text{ such that } a \leq a'\}. \tag{2.6}$$

Huang (2002, Theorem 3.2) shows that if $x \in X(\mathcal{F}^i)$, then $\pi_t(x;\mathcal{F}^i)$ represents also the minimum replication cost at period t of the dividend stream $(x_s)_{s>t}$ for agent i, when he is restricted to holding trading strategies in $R(\Theta(p;\mathcal{F}^i))$:

$$\pi_t(x;\mathcal{F}^i) = \inf\{p_t\theta_t \mid \theta \in R(\Theta(p;\mathcal{F}^i)), (p_s + d_s)\theta_{s-1} - p_s\theta_s \geq x_s, \forall s > t\}. \tag{2.7}$$

I illustrate the concepts introduced in this section for the most common portfolio constraints encountered in the literature (see, for example, He and Modest 1995): borrowing constraints, solvency (debt) constraints and short sale constraints.

2.2.1 Borrowing constraints

Agent i is subject to *borrowing* limits $w^i \in X(\mathcal{F}^i)$ if[9]

$$\Theta(p;\mathcal{F}^i) = \left\{\theta \in X^J(\mathcal{F}^i) \mid p_t\theta_t \geq w^i_t, \forall t \geq 0\right\}. \tag{2.8}$$

I assume that $-w^i \in X_+(\mathcal{F}^i)$ ($w^i \leq 0$), thus agents are not subjected to forced saving. Notice that

$$R(\Theta(p;\mathcal{F}^i)) = \{\theta \in X^J(\mathcal{F}^i) \mid p_t\theta_t \geq 0, \forall t \geq 0\}, \tag{2.9}$$

therefore for each $\omega \in \Omega$ and $t \in \mathbb{N}$,

$$\left\{\theta_t(\omega) \mid \theta \in R(\Theta(p;\mathcal{F}^i))\right\} = \{z \in \mathbb{R}^J \mid p_t(\omega)z \geq 0\},$$

which is a polyhedral cone in $\mathbb{R}^J$ containing $\mathbb{R}^J_+$. Motzkin's (1951) transposition theorem and (2.3) and (2.9) imply that

$$A(p;\mathcal{F}^i) = \Big\{a \in \bar{X}_{++}(\mathcal{F}^i) \mid a_0 = 1 \text{ and there is a } \phi \in X_+(\mathcal{F}^i) \text{ such that } p_t - E_{\mathcal{F}^i_t}\frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1}) = \phi_t p_t, \forall t \geq 0\Big\}. \tag{2.10}$$

[9]Borrowing constraints are referred to as *wealth* constraints by Santos and Woodford (1997). I avoid this terminology, as both borrowing and solvency (debt) constraints (studied in the next section) impose bounds on end of period, respectively beginning of period wealth, and therefore can be regarded as wealth constraints.

Notice that agent i's marginal utilities belongs to $A(p;\mathcal{F}^i)$ since the first order conditions for agent i at t are (He and Modest 1995)

$$p_t - E_{\mathcal{F}_t^i} \frac{u_{t+1}^{i\prime}(c_{t+1}^i)}{u_t^{i\prime}(c_t^i)}(p_{t+1} + d_{t+1}) = \phi_t p_t,$$

where $\phi_t \geq 0$ is a Kuhn-Tucker multiplier.

The set of maximal state price densities $\bar{A}(p;\mathcal{F}^i)$ is therefore equal to $A^{=}(p;\mathcal{F}^i)$, the set of processes that *martingale-price* all assets:

$$A^{=}(p;\mathcal{F}^i) := \left\{ a \in X_{++}(\mathcal{F}^i) \mid a_0 = 1,\ p_t = E_{\mathcal{F}_t^i} \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1}), \forall t \geq 0 \right\}. \quad (2.11)$$

2.2.2 Solvency (debt) constraints

Solvency constraints (He and Modest 1995, Alvarez and Jermann 2000) are also referred to as *debt* constraints (Hellwig and Lorenzoni 2009). I will use the two terms interchangeably. Agent i is subject to *solvency (debt) limits* $w^i \in X(\mathcal{F}^i)$ if

$$\Theta(p;\mathcal{F}^i) = \left\{ \theta \in X^J(\mathcal{F}^i) \mid (p_{t+1} + d_{t+1})\theta_t \geq w_{t+1}^i, \forall t \geq 0 \right\}. \quad (2.12)$$

Debt limits are assumed to be nonpositive, that is $w^i \leq 0$. Notice that

$$R(\Theta(p;\mathcal{F}^i)) = \{\theta \in X^J(\mathcal{F}^i) \mid (p_{t+1} + d_{t+1})\theta_t \geq 0, \forall t \geq 0\}, \quad (2.13)$$

therefore for each $\omega \in \Omega$ and $t \in \mathbb{N}$, $\{\theta_t(\omega) \mid \theta \in R(\Theta(p;\mathcal{F}^i))\}$ contains $\mathbb{R}_+^J$ and is a polyhedral cone, as it is an intersection of a finite number of half-spaces.[10] It follows that for all $t \geq 0$ and $\theta \in X^J$,

$$\left(p_t - E_{\mathcal{F}_t^i} \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1}) \right) \theta_t \geq 0 \text{ if } (p_{t+1} + d_{t+1})\theta_t \geq 0. \quad (2.14)$$

[10] This number equals the number of tree branches originating from $\mathcal{F}_t(\omega)$, or the number of atoms of $\mathcal{F}_{t+1}$ that are included in $\mathcal{F}_t(\omega)$.

Motzkin's (1951) transposition theorem and (2.14) imply that

$$A(p;\mathcal{F}^i) = \big\{a \in \bar{X}_{++}(\mathcal{F}^i) \mid a_0 = 1 \text{ and there exists } \phi \in X_+(\mathcal{F}^i) \text{ such that} \\ p_t = E_{\mathcal{F}^i_t}\left((a_{t+1}/a_t + \phi_{t+1})\,(p_{t+1} + d_{t+1})\right)\big\}\,.$$

Clearly $\bar{A}(p;\mathcal{F}^i)$ is equal to the set of martingale-pricing densities $A^{=}(p;\mathcal{F}^i)$ in (2.11).

Again agent i's marginal utilities belongs to $A(p;\mathcal{F}^i)$ since the first order conditions at t are (He and Modest 1995):

$$p_t = E_{\mathcal{F}^i_t}(u^{i\prime}_{t+1}(c^i_{t+1})/u^{i\prime}_t(c^i_t) + \phi_{t+1})(p_{t+1} + d_{t+1}),$$

where $\phi_{t+1} \geq 0$ is a ($\mathcal{F}^i_{t+1}$-measurable) Kuhn-Tucker multiplier.

2.2.3 Short sale constraints

For each agent i, let $w^i \in X^J(\mathcal{F}^i)$ with $w^i \leq 0$. Agent i is subject to *short sale limits* w^i if

$$\Theta(p;\mathcal{F}^i) = \big\{\theta \in X^J(\mathcal{F}^i) \mid \theta_t \geq w^i_t, \forall t \geq 0\big\}\,. \tag{2.15}$$

In this case,

$$R(\Theta(p;\mathcal{F}^i)) = X^J_+(\mathcal{F}^i), \tag{2.16}$$

and therefore for each $\omega \in \Omega$ and $t \in \mathbb{N}$, $\{\theta_t(\omega) \mid \theta \in R(\Theta(p;\mathcal{F}^i))\}$ is indeed a polyhedral cone, equal to $\mathbb{R}^J_+$. By (2.3) and (2.16),

$$A(p;\mathcal{F}^i) = \left\{a \in X_{++}(\mathcal{F}^i) \mid a_0 = 1,\ p_t \geq E_{\mathcal{F}^i_t}\frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1}), \forall t \geq 0\right\}. \tag{2.17}$$

Agent i's marginal utilities are a valid state price density, since the first order conditions are (He and Modest 1995)

$$p_t - E_t\frac{u^{i\prime}_{t+1}(c^i_{t+1})}{u^{i\prime}_t(c^i_t)}(p_{t+1} + d_{t+1}) \geq 0, \forall t \geq 0. \tag{2.18}$$

2.3 Price decomposition

The state price densities constructed using the absence of unrestricted arbitrage opportunities are used here to define bubbles. Fix an $a \in \bar{A}(p; \mathcal{F}^i)$. Assumption 2.1 and (2.3) ensure that

$$p_t - E_{\mathcal{F}_t^i} \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1}) \geq 0, \forall t \geq 0. \tag{2.19}$$

Define the (vector) process $m(a, p; \mathcal{F}^i) \in X_+^J(\mathcal{F}^i)$ as

$$m_t(a, p; \mathcal{F}^i) := p_t - E_{\mathcal{F}_t^i} \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1}) \quad (\geq 0), \forall t \geq 0. \tag{2.20}$$

It follows that

$$p_t = \frac{1}{a_t} E_{\mathcal{F}_t^i} \sum_{s=t+1}^{T} a_s d_s + \frac{1}{a_t} E_{\mathcal{F}_t^i} \sum_{s=t}^{T-1} a_s m_s(a, p; \mathcal{F}^i) + \frac{1}{a_t} E_{\mathcal{F}_t^i} a_T p_T, \tag{2.21}$$

and therefore

$$p_t = \frac{1}{a_t} E_{\mathcal{F}_t^i} \sum_{s>t} a_s d_s + \frac{1}{a_t} E_{\mathcal{F}_t^i} \sum_{s \geq t} a_s m_s(a, p; \mathcal{F}^i) + \lim_{T \to \infty} \frac{1}{a_t} E_{\mathcal{F}_t^i} a_T p_T. \tag{2.22}$$

Let

$$b_t(a, p; \mathcal{F}^i) := \lim_{T \to \infty} \frac{1}{a_t} E_{\mathcal{F}_t^i} a_T p_T. \tag{2.23}$$

As a consequence of (2.22), the process $b = (b^1, \ldots, b^J)$ is well-defined, non-negative, and $a \cdot b(a, p; \mathcal{F}^i)$ is a martingale.

Therefore the price of each asset $j \in \{1, \ldots, J\}$ at t can be decomposed into three nonnegative terms:

$$p_t^j = \underbrace{\frac{1}{a_t} E_{\mathcal{F}_t^i} \sum_{s>t} a_s d_s^j}_{\text{dpv of dividends}} + \underbrace{\frac{1}{a_t} E_{\mathcal{F}_t^i} \sum_{s \geq t} a_s m_s^j(a, p; \mathcal{F}^i)}_{\text{resale option (convenience yield)}} + \underbrace{\lim_{T \to \infty} \frac{1}{a_t} E_{\mathcal{F}_t^i} a_T p_T^j}_{\text{bubble } b_t^j(a,p;\mathcal{F}^i)}. \tag{2.24}$$

The first term is $f_t(a, d^j; \mathcal{F}^i)$, which shows that the price of the asset is at least as high as the discounted present value of its dividends, using as discount rates any state price density associated to the agent.

The last term $b_t^j(a, p; \mathcal{F}^i)$ is referred to as a *bubble* in asset j under the state price density a, whenever it is different from zero. Since $a \cdot b^j(a, p; \mathcal{F}^i)$ is a martingale, if $b_t^j(a, p; \mathcal{F}^i) > 0$ then it must be the case that $b_0^j(a, p; \mathcal{F}^i) > 0$. Following the terminology of Santos and Woodford (1997), we say that asset j *unambiguously (ambiguously)* has a *bubble* at t if $b_t^j(a, p; \mathcal{F}^i) > 0$ for all (some) $a \in A(p; \mathcal{F}^i)$ and for all (some) agents i. Thus the price of an asset having an unambiguous (ambiguous) bubble component exceeds the valuations of the asset dividend for all (some) agents and for all (some) agent specific state price densities.

The middle term $\frac{1}{a_t} E_{\mathcal{F}_t^i} \sum_{s \geq t} a_s m_s(a, p^j; \mathcal{F}^i)$ in the decomposition (2.24) represents the *resale option* of agent i from being long a unit of asset j. The term was coined by Scheinkman and Xiong (2003). If the state price density equals the marginal utility of the agent i, that is if $a_t/a_0 = u_t^{i\prime}(c_t^i)/u_0^{i\prime}(c_0^i)$,[11] then the discounted present value of dividends $\frac{1}{a_t} E_{\mathcal{F}_t^i} \sum_{s>t} a_s d_s^j$ represents what an agent is willing to pay if he was forced to maintain the holdings of the asset forever. Thus the resale option is the portion of the asset price that can be attributed to agent's ability to sell the asset in the future to unconstrained agents having a higher valuation of the asset.

Pascoa, Petrassi, and Torres-Martinez (2011) and Araujo, Páscoa, and Torres-Martínez (2011) also emphasize that the resale option is more appropriately interpreted as the relaxation of the financial constraints, rather than a "bubble". It represents the present value of future *services* in relaxing financial constraints, or equivalently, the present value of *shadow prices* of the constraints. Finally, the resale option can be interpreted as the *convenience yield* generated by being long one unit of the asset j over its lifetime. With shorting restrictions, the holding of inventories gives the owner of the asset the possibility to better smooth demand shocks by selling the asset, especially when it commands a high price, and gives rise to a convenience yield (Cochrane 2002).

[11] The division by $u_0^{i\prime}(c_0^i)$ insures that the normalization $a_0 = 1$ holds. As seen in Section 2.2, each agent's marginal utilities are a valid state price density for borrowing, debt and short sale constraints.

Under (martingale-pricing) state price densities in $A^{=}(p;\mathcal{F}^i)$ (see (2.11)), the resale option is zero. Therefore our definition of bubbles given for general portfolio constraints and differential information reduces to the definition employed by Santos and Woodford (1997) for economies with borrowing constraints and symmetric information ($P^i = P$, $\mathcal{F}^i = \mathcal{F}$ for all i).

3 Non-existence of bubbles

This section gives sufficient conditions under which bubbles on assets in positive supply cannot be present in environments with *high interest rates*. Formally, interest (discount) rates are *high* from the point of view of an agent i if $\pi_0(e;\mathcal{F}^i) < \infty$, that is, if the discounted present value of the aggregate endowment is finite under all agent's i state price densities. Equivalently, agent i has a trading strategy in the recession cone of his constraints whose payoffs dominate (from above) the aggregate endowment. Interest rates are said to be *high* if they are high from the point of view of each agent.

The first result (Theorem 3.1) shows that if interest rates are high from the point of view of an uninformed agent (having only public information), then there exists a state price density associated to that agent under which there is no bubble for all assets in positive supply.

Theorem 3.1. *Consider an equilibrium $\big(p,(c^i)_{i=1}^I,(\theta^i)_{i=1}^I\big)$. Assume that there exists an uninformed agent $k \in \{1,\dots,I\}$, (that is, with $\mathcal{F}^k = \mathcal{F}^m$) with portfolio constraints satisfying $R(\Theta(p;\mathcal{F}^k)) \subset R(\Theta(p;\mathcal{F}^i))$ for all $i \in \{1,\dots,I\}$. If $\pi_0(e;\mathcal{F}^k) < \infty$, there exists $a^k \in \bar{A}(p;\mathcal{F}^k)$ such that $b_0^j(a^k,p;\mathcal{F}^k) = 0$ for all securities j in positive supply.*

Proof. Since $d \in X^J(\mathcal{F}^k)$ and $e \in X_+(\mathcal{F}^k)$ it follows that $\tilde{e} \in X_+(\mathcal{F}^k)$. As buy and hold portfolios of each asset are feasible, (2.7) implies that $\pi_t(d^j;\mathcal{F}^k) < p_t^j < \infty$, for all $t \in \mathbb{N}$ and $j \in J$. Therefore $\pi_t(\tilde{e};\mathcal{F}^k) < \infty$. For each $\varepsilon > 0$ and $t \in \mathbb{N}$, (2.5) shows that there exists $T(\varepsilon, t) \in \mathbb{N}$ and $a^k(\varepsilon, t) \in \bar{A}(p; \mathcal{F}^k)$ such that for all $T \geq T(\varepsilon, t)$,

$$\frac{1}{a_t^k(\varepsilon, t)} E_{\mathcal{F}_t^k} \sum_{s=t+1}^{T} a_s^k(\varepsilon, t)\tilde{e}_s \geq \pi_t(\tilde{e}; \mathcal{F}^k) - \varepsilon \geq$$

$$\geq \frac{1}{a_t^k(\varepsilon, t)} E_{\mathcal{F}_t^k} \left(\sum_{s=t+1}^{T} a_s^k(\varepsilon, t)\tilde{e}_s + a_T^k(\varepsilon, t)\pi_T(\tilde{e}; \mathcal{F}^k) \right) - \varepsilon.$$

It follows that

$$\frac{1}{a_t^k(\varepsilon, t)} E_{\mathcal{F}_t^k} a_T^k(\varepsilon, t)\pi_T(\tilde{e}; \mathcal{F}^k) \leq \varepsilon, \forall T \geq T(\varepsilon, t). \tag{3.1}$$

Fix an $\varepsilon_0 > 0$. Using (3.1), we construct inductively a sequence (T_n) and an $a^k \in \bar{A}(p; \mathcal{F}^k)$ with $a_0^k = 1$, $T_0 = 0$ and such that

$$E_{\mathcal{F}_{T_n}^k} a_{T_{n+1}}^k \pi_{T_{n+1}}(\tilde{e}; \mathcal{F}^k) \leq \varepsilon_0 / 2^n, \forall n \geq 0. \tag{3.2}$$

Notice that $c^i \leq e^i + (c^i - e^i)^+$, where $(c^i - e^i)^+ := \max\{c^i - e^i, 0\}$. As $R(\Theta(p; \mathcal{F}^k)) \subset R(\Theta(p; \mathcal{F}^i))$, it follows by (2.7) that

$$\pi_s((c^i - e^i)^+; \mathcal{F}^i) < \pi_s(\tilde{e}; \mathcal{F}^i) < \pi_s(\tilde{e}; \mathcal{F}^k) < \infty, \forall s. \tag{3.3}$$

The optimality of (c^i, θ^i) implies that for each agent i,

$$p_s\theta_s^i \leq \pi_s((c^i - e^i)^+; \mathcal{F}^i), \forall s \geq 0. \tag{3.4}$$

Indeed, assume by contradiction that $p_s\theta_s^i > \pi_s((c^i - e^i)^+; \mathcal{F}^i)$. Therefore there exists a $\theta \in R(\Theta(p; \mathcal{F}^i))$ super-replicating $((c_n^i - e_n^i)^+)_{n \geq s+1}$ and such that

$$p_s\theta_s^i > p_s\theta_s \geq \pi_s((c^i - e^i)^+; \mathcal{F}^i).$$

Since $R(\Theta(p; \mathcal{F}^i)) \subset \Theta(p; \mathcal{F}^i)$, agent i can switch at s to strategy θ instead of θ^i, which is feasible and leads to a higher consumption for the agent in all future periods and strictly higher at period s, contradicting the optimality of c^i.

Using (3.3),

$$p_s\bar{\theta} = \sum_i p_s\theta_s^i \leq \sum_i \pi_s^i((c^i - e^i)^+; \mathcal{F}^i) \leq \sum_i \pi_s(\tilde{e}; \mathcal{F}^i) \leq I\pi_s(\tilde{e}; \mathcal{F}^k), \forall s \geq 0. \quad (3.5)$$

By (3.2) and (3.5),

$$E_{\mathcal{F}_0^k} a_{T_n}^k p_{T_n}\bar{\theta} \leq I \cdot E_{\mathcal{F}_0^k} a_{T_n}^k \cdot \pi_{T_n}(\tilde{e}; \mathcal{F}^k) \leq I \cdot \varepsilon_0/2^{n-1}, \forall n \geq 1.$$

The conclusion follows, as

$$b_0(a^k, p; \mathcal{F}^k) \cdot \bar{\theta} = \lim_{n\to\infty} E_{\mathcal{F}_0^k} a_{T_n}^k p_{T_n}\bar{\theta} = 0.$$

□

The theorem shows that assets in positive supply have no (unambiguous) bubbles from the point of view of all uninformed agents with high interest rates. The assumption $R(\Theta(p;\mathcal{F}^k)) \subset R(\Theta(p;\mathcal{F}^i))$ for all $i \in \{1,\ldots,I\}$ made on the portfolio constraints is mild and is satisfied whenever agents with coarser information have access to fewer unrestricted portfolios. It holds, in particular, for borrowing, debt and short sale constraints.

Theorem 3.1 in Santos and Woodford (1997) obtains as a particular case of the theorem above, by assuming symmetric information and homogeneous beliefs ($\mathcal{F}^i = \mathcal{F}$, $P^i = P$, for all i) and portfolio restrictions in the form of borrowing constraints.

When agents are subject to *no short sales* constraints, that is when they face short sale limits $w^i = 0$ (see Section 2.2.3), the existence of an uninformed agent can be replaced by the assumption that there exists an agent with high interest rates who is (uniformly) unconstrained infinitely often. Agent i is unconstrained infinitely often in asset j if there exists $\varepsilon > 0$ such that the event $\{\theta_t^{i,j} \geq \varepsilon\}$ occurs infinitely often, that is $P\left(\cap_{s=0}^{\infty} \cup_{t\geq s} \{\theta_t^{i,j} \geq \varepsilon\}\right) = 1$.

Proposition 3.2. *Assume that no short sales are allowed and let j be an asset in positive supply. If agent i has high interest rates ($\pi_0(e;\mathcal{F}^i) < \infty$) and if he* *is unconstrained in j infinitely often, then there exists $a^i \in \bar{A}(p; \mathcal{F}^i)$ such that $b_0^j(a^i, p; \mathcal{F}^i) = 0$.*

Proof. Let $\varepsilon > 0$ such that $\{\theta_t^{i,j} \geq \varepsilon\}$ occurs infinitely often, and let $\varepsilon_0 \in (0, \varepsilon)$ arbitrary. By repeating the arguments in the proof of Theorem 3.1, there exists an increasing sequence $(T_n)_{n=1}^{\infty}$ such that for all $n \geq 0$,

$$E_{\mathcal{F}^i_{T_n}} a^i_{T_{n+1}} \pi_{T_{n+1}}(\tilde{e}; \mathcal{F}^i) \leq \varepsilon_0 / 2^n \text{ and } \theta^{i,j}_{T_n} > \varepsilon.$$

Using (3.4),

$$p^j_{T_n} \cdot \varepsilon \leq p^j_{T_n} \theta^{i,j}_{T_n} \leq p_{T_n} \theta^i_{T_n} \leq \pi_{T_n}((c^i - e^i)^+; \mathcal{F}^i) \leq \pi_{T_n}(\tilde{e}; \mathcal{F}^i), \ \forall n \geq 1.$$

Therefore

$$E_{\mathcal{F}^i_0} a^i_{T_n} p^j_{T_n} \leq \frac{\varepsilon_0}{\varepsilon} \cdot \frac{1}{2^{n-1}}, \forall n \geq 1,$$

and the conclusion follows by letting $n \to \infty$. □

For a deterministic economy, Proposition 3.2 shows that bubbles in a positive supply asset j can exist only if $\liminf_{t \to \infty} \theta_t^{i,j} = 0$, for all agents i, which is Proposition 3 in Kocherlakota (1992).

For debt and borrowing constraints, $R(\Theta(p; \mathcal{F}^i))$ depends only on the filtration $\mathcal{F}^i$ and not on the actual bounds w^i faced by the agent i. Therefore $R(\Theta(p; \mathcal{F}^m))$ can be defined by extension, replacing $\mathcal{F}^i$ by $\mathcal{F}^m$ in (2.9) or (2.13). Hence $\bar{A}(p; \mathcal{F}^m) = A^{=}(p; \mathcal{F}^m)$ and $\pi_0(e; \mathcal{F}^m)$ are also well-defined. Even without assuming the existence of uninformed agents, Theorem 3.1 implies that if interest rates are *high* with respect to *public information* (that is, if $\pi_0(e; \mathcal{F}^m) < \infty$), there exists a "common-knowledge" state price density $a^m \in A(p; \mathcal{F}^m)$ such that there are no bubbles on assets in positive supply under a^m. Each agent specific state price density $a^i \in A(p; \mathcal{F}^i)$ leads to a common knowledge state price density if "averaged" over the public information. If markets are complete with respect to the public information $\mathcal{F}^m$, then there is a unique common-knowledge state price density. This implies, in turn, that there cannot be bubbles under any agent specific state price density, and therefore even *ambiguous* bubbles are absent. This reasoning is captured in Proposition 3.3 below. Formally, markets are complete with respect to the public information $\mathcal{F}^m$ if for any $t \geq 0$ and any $\mathcal{F}^m_{t+1}$-measurable random variable x_{t+1}, there exists an $\mathcal{F}^m_t$-measurable J-dimensional random vector θ_t such that $(p_{t+1} + d_{t+1})\theta_t = x_{t+1}$. The definition captures the idea that, ignoring portfolio constraints, an agent with information $\mathcal{F}^m$ is able to achieve any desired payoff in all next period contingencies that he thinks can arise, by trading in the available assets.

Proposition 3.3. *Assume that all agents are subject to either debt or borrowing constraints and that markets are complete and interest rates are high with respect to the public information. Then* $b^j_0(a^i, p; \mathcal{F}^i) = 0$ *for each* $i \in \{1, \ldots, I\}$*, each* $a^i \in \bar{A}(p; \mathcal{F}^i)$ *and each security* j *in positive supply.*

Proof. Market completeness implies that $A(p; \mathcal{F}^m) = A^{=}(p; \mathcal{F}^m) = \{\bar{a}^m\}$ (singleton). With an identical proof as in Theorem 3.1, $b^j_t(\bar{a}^m, p; \mathcal{F}^m) = 0$ for all securities j in positive supply and all dates t.

Let $a^i \in \bar{A}(p; \mathcal{F}^i)$ $(= A^{=}(p; \mathcal{F}^i))$ and construct $a^m \in \bar{X}(\mathcal{F}^m)$ as $a^m_t := E_{\mathcal{F}^m_t} a^i_t$. By the pull-out and chain rule properties of conditional expectation (Kallenberg 2002, Theorem 5.1),

$$a^m_t p_t = E_{\mathcal{F}^m_t} a^i_t p_t = E_{\mathcal{F}^m_t} E_{\mathcal{F}^i_t} a^i_{t+1}(p_{t+1} + d_{t+1}) =$$
$$= E_{\mathcal{F}^m_t}(p_{t+1} + d_{t+1}) E_{\mathcal{F}^m_{t+1}} a^i_{t+1} = E_{\mathcal{F}^m_t} a^m_{t+1}(p_{t+1} + d_{t+1}),$$

thus $a^m \in A^{=}(p; \mathcal{F}^m)$ and $a^m = \bar{a}^m$. For any security j in positive supply,

$$E_{\mathcal{F}^m_t}\left(a^i_t \cdot b^j_t(a^i, p; \mathcal{F}^i)\right) = E_{\mathcal{F}^m_t} \lim_{T\to\infty} E_{\mathcal{F}^i_t} a^i_T p^j_T = \lim_{T\to\infty} E_{\mathcal{F}^m_t} E_{\mathcal{F}^i_t} a^i_T p^j_T =$$
$$= \lim_{T\to\infty} E_{\mathcal{F}^m_t} p^j_T E_{F^m_T} a^i_T = a^m_t b^j_t(a^m, p; \mathcal{F}^m) = \bar{a}^m_t b^j_t(\bar{a}^m, p; \mathcal{F}^m) = 0, \qquad (3.6)$$

where the first equality is the definition of the bubble, the second holds by the Lebesgue dominated convergence theorem, since $E_{\mathcal{F}^i_t} a^i_T p^j_T \leq a^i_t p_t$ (by (2.24)), and the rest follow from the pull-out and chain rule properties of conditional expectation. Therefore $b^j_t(a^i, p; \mathcal{F}^i) = 0$ for any date t, agent i, state price density $a^i \in \bar{A}(p, d; F^i)$

and asset j in positive supply. □

Proposition 3.3 gives conditions under which the price of all assets in positive supply equal the discounted present value of their dividends (as resale options are zero) under all state price densities, for all agents.

All the non-existence of bubbles results presented so far are somewhat restrictive. Theorem 3.1 ensures that there are no bubbles from the point of view of uninformed agents with high interest rates. This limits severely the scope of the theorem, if there are no (or few relative to the size of the economy) uninformed agents. Proposition 3.2 applies only to the case of no short sale constraints. Both results rule out only unambiguous bubbles. Thus an agent might perceive that there is a bubble under a state price density but not under another. Moreover, some agents think that there are bubbles while others don't. Examples of ambiguous bubbles can be constructed even under symmetric information (Santos and Woodford 1997). Proposition 3.3 rules out also ambiguous bubbles, but it requires complete markets with respect to the public information.

By strengthening the assumptions on agents' preferences and imposing a form of impatience, ambiguous bubbles can also be excluded. The non-existence of bubbles applies to economies with general portfolio constraints and the presence of uninformed agents or market completeness with respect to public information is not needed. It will be shown that the bubble component in a positive supply asset is zero under any state price density of an agent with high interest rates (Theorem 3.4).

The following impatience assumption requires that at any date and state, an agent prefers adding a given multiple of the aggregate supply of goods in the economy to his current consumption at the cost of reducing all his future consumption by a small fraction. It is satisfied by any continuous, stationary, recursive utility that discounts the future (Santos and Woodford 1997), and holds trivially in finite horizon economies. It was used also in the literature dealing with the existence of equilibrium in infinite horizon economies (Levine and Zame 1996, Magill and Quinzii 1994).

Assumption 3.1. *For each agent i, there exists $\kappa^i > 0$ and $\eta^i \in (0,1)$ such that for any date t and any consumption process $c \leq \tilde{e}$, $U^i_t(c_t + \kappa^i \tilde{e}_t, \eta^i c_{t+1}, \eta^i c_{t+2}, \dots) > U^i_t(c)$.*

We also assume that the portfolio constraints do not prevent agents from scaling back slightly their portfolios if they choose to do so.

Assumption 3.2. *For each agent i, there exists $\lambda^i \in (0,1)$ such that for any $\theta^i \in \Theta(p;\mathcal{F}^i)$, any $1 > \lambda \geq \lambda^i$ and any date t, $\lambda\theta^i_s \in \Theta_s(p;\mathcal{F}^i)$ for all $s \geq t$.*

Borrowing, debt and short sale constraints satisfy Assumption 3.2, as $w^i \leq 0$.

Theorem 3.4. *Consider an equilibrium $\big(p,(c^i)_{i=1}^I,(\theta^i)_{i=1}^I\big)$. Suppose that Assumptions 3.1 and 3.2 hold. If $\pi_t(e;\mathcal{F}^n) < \infty$ for some $n \in \{1,\ldots,I\}$, then for any $a \in A(p;\mathcal{F}^n)$, $b^j_t(a,d;\mathcal{F}^n) = 0$ for all securities j in positive supply.*

Proof. For each agent i, let $\gamma^i := \max\{\eta^i,\lambda^i\}$. Assumptions 3.1 and 3.2 imply that

$$(1-\gamma^i)p_s\theta^i_s \leq \kappa^i\tilde{e}_s, \forall s \in \mathbb{N}. \tag{3.7}$$

Indeed, suppose by contradiction that there exists $s \in \mathbb{N}$ such that $(1-\gamma^i)p_s\theta^i_s > \kappa^i\tilde{e}_s$ with positive probability. Construct the alternative consumption and trading strategies

$$\hat{c}^i := (c^i_0,\ldots,c^i_{s-1},c^i_s+(1-\gamma^i)p_s\theta^i_s,\gamma^i c^i_{s+1},\gamma^i c^i_{s+2},\ldots),$$
$$\hat{\theta}^i := (\theta^i_0,\ldots,\theta^i_{s-1},\gamma^i\theta^i_s,\gamma^i\theta^i_{s+1},\ldots).$$

Notice that $\hat{\theta}^i \in \Theta(p;\mathcal{F}^i)$, by Assumption 3.2. Moreover, $(\hat{c}^i,\hat{\theta}^i) \in B^i(p)$, and $U^i_s(\hat{c}^i) > U^i_s(c^i)$ on the set $\{(1-\gamma^i)p_s\theta^i_s > \kappa^i\tilde{e}_s\}$. Therefore agent i can sell at s in states $\{(1-\gamma^i)p_s\theta^i_s > \kappa^i\tilde{e}_s\}$ a fraction $1-\gamma^i$ of his portfolio θ^i_s and increase his consumption at s by more than $k^i\tilde{e}_s$ while consuming, respectively holding, a fraction γ^i of the initial consumption, respectively portfolios, for all periods greater than t. This would strictly increase his utility, contradicting the optimality of (c^i,θ^i) for agent i. It follows that

$$(1-\gamma)p_s\bar{\theta} = (1-\gamma)p_s\sum_i \theta^i_s \leq \sum_i \kappa^i\tilde{e}_s \leq I\kappa\tilde{e}_s, \forall s \geq 0, \tag{3.8}$$

where $\gamma := \max_i \gamma^i$ and $\kappa := \max_i \kappa^i$. Hence for any $a \in A(p; \mathcal{F}^n)$,

$$\frac{1}{a_t} E_{\mathcal{F}_t^n} a_s p_s \bar{\theta} \leq \frac{I\kappa}{1-\gamma} \frac{1}{a_t} E_{\mathcal{F}_t^n} a_s \tilde{e}_s, \forall s \geq t. \tag{3.9}$$

Since $\pi_t(e; \mathcal{F}^n) < \infty$,

$$\frac{1}{a_t} E_{\mathcal{F}_t^n} \sum_{s>t} a_s \tilde{e}_s \leq \pi_t(\tilde{e}; \mathcal{F}^n) < \infty, \tag{3.10}$$

and thus $\lim_{s\to\infty} E_{\mathcal{F}_t^n} a_s p_s \bar{\theta} \leq \frac{I\kappa}{1-\gamma} \lim_{s\to\infty} E_{\mathcal{F}_t^n} a_s \tilde{e}_s = 0$. The conclusion follows. $\square$

This result extends Theorem 3.3 of Santos and Woodford (1997) to asymmetric information economies and general portfolio constraints. It was obtained also by Yu (1998) for the particular case of borrowing constraints and homogeneous beliefs.

The absence of bubbles on an asset under a martingale-pricing state price density (for example, for borrowing or debt constraints) is tantamount to having the price of the asset equal to its fundamental value under that state price density. State price densities that jointly martingale-price the assets might not exist for short sale constraints, and the price of an asset might not, in principle, equal the discounted present value of its dividends, due to the presence of a resale option (convenience yield).

A large body of literature, initiated by Harrison and Kreps (1978) and reviewed in the introduction, analyzes the resale options (called "speculative bubbles") in economies with short sales constraints and heterogeneous beliefs (but symmetric information), where the only allowable state price densities are those equal to the marginal utilities of the agents.

For economies with short sale constraints and heterogeneous beliefs, the intertemporal marginal rate of substitutions of the *unconstrained* agents acquiring the asset (being long) at each period of time lead to valid state price densities. Under such discount factors, the resale option disappears. Combining this observation with Theorem 3.4 on non-existence of bubbles, Proposition 3.5 shows that the price of an asset in positive supply cannot be unambiguously higher than the discounted present value of its dividends, even with heterogeneous beliefs and short sale constraints (but no asymmetric information).

Proposition 3.5. *Consider an equilibrium* $\left(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ *in which agents are subject to short sale constraints. Assume that asset* j *is in positive supply. If* $\{\pi_t(e; \mathcal{F}) < \infty\}$ *and Assumptions 3.1 and 3.2 hold, then there exists* $a \in A(p; \mathcal{F})$ *such that* $p_t^j = f(a, d^j; \mathcal{F})$.

Proof. Agent i's first order conditions are given by (2.18) and the complementary slackness condition

$$\left(p_t - E_{\mathcal{F}_t} \frac{u_{t+1}^{i\prime}(c_{t+1}^i)}{u_t^{i\prime}(c_t^i)}(p_{t+1} + d_{t+1})\right)(\theta_t^i - w_t^i) = 0, \forall t \geq 0.$$

Define the process $\imath$ with values in $\{1, \ldots, I\}$ representing the "unconstrained" agent holding positive amounts of asset j at each date and state:

$$\imath_t(\omega) := \min\{i \mid i \in \{1, \ldots, I\}, \theta_t^{i,j}(\omega) > w_t^{i,j}(\omega)\}, \forall t \in \mathbb{N}, \forall \omega \in \Omega.$$

The information of the unconstrained agent $\imath$ is $\mathcal{F}^\imath \subset \mathcal{F}$ given by $\mathcal{F}_t^\imath(\omega) := \mathcal{F}_t^{\imath_t(\omega)}(\omega)$, for all t, ω. By construction, the resale option on asset j is zero when the intertemporal marginal rates of substitution of the unconstrained agent $\imath$ are used in discounting, in other words when discounting is done with $a \in X_{++}(\mathcal{F})$ constructed as $a_0 := 1$ and

$$a_{t+1} := a_0 \cdot \prod_{n=0}^{t} \frac{u_{n+1}^{\imath_n\prime}(c_{n+1}^{\imath_n})}{u_n^{\imath_n\prime}(c_n^{\imath_n})}, \forall t \geq 0.$$

Moreover, by Theorem 3.4 applied to $\mathcal{F}$ rather than $\mathcal{F}^n$, there are no bubbles in j under state price density a, and therefore the price of the asset j equals the discounted present value of its dividends. □

In the previous Proposition, $R(\Theta(p; \mathcal{F}))$, $A(p; \mathcal{F})$ and $\pi_t(e; \mathcal{F})$ are defined by extension using (2.16), (2.17) and (2.5) (or (2.7)) with $\mathcal{F}^i$ substituted by $\mathcal{F}$. With symmetric information ($\mathcal{F}^i = \mathcal{F} = \mathcal{F}^m$), all agents have an identical set of state price densities ($A(p; \mathcal{F}^i) = A(p; \mathcal{F})$) and therefore Proposition 3.5 guarantees that for each asset in positive supply, there exists a state price density common to all agents under which the resale option component and the bubble component are zero.

With asymmetric information ($\mathcal{F}^i \neq \mathcal{F}^m$), resale options can be positive under any state price density $a \in A(p; \mathcal{F}^i)$, for all agents i. Such examples with positive resale options are constructed in Allen, Morris, and Postlewaite (1993). The reason is that the unconstrained agent $\imath$ (see the construction in the proof of Proposition 3.5) pools the information of several agents. Formally, the filtration $\mathcal{F}^\imath$ of the unconstrained agent $\imath$ does not coincide with the filtration of some agent i, thus $\mathcal{F}^\imath \neq \mathcal{F}^i$, for all $i \in \{1, \ldots, I\}$. With asymmetric information, Proposition 3.5 only shows that all fully informed agents (if they exist) can rationalize the price of the asset as being equal to the present value of its dividends, if they have high interest rates.

4 Conclusion

I show that the non-existence of bubbles in economies with high interest rates (Santos and Woodford 1997) extends also to economies with general portfolio constraints (rather than just borrowing constraints) and with differential information (heterogeneous beliefs and asymmetric information).

With short sale constraints and heterogeneous beliefs, but no asymmetric information, (unambiguous) resale options cannot exist. With asymmetric information, resale options are possible. The scope of asymmetric information in generating resale options is intrinsically limited by the revelation of information through prices in rational expectations equilibria. For full revelation not to obtain generically, the number of prices has to be smaller than the number of sources of uncertainty (Allen and Jordan 1998).

To my knowledge, there are no known generic full revelation results for the actual infinite horizon model employed here. Moreover, as explained in Allen, Morris, and Postlewaite (1993), some of the states in the probability space are meant to capture the uncertainty about what other agents know, rather than uncertainty about fundamentals. Perturbations of the parameters that do not affect fundamentals in those "private" states will not lead (generically) to distinct prices and therefore full revelation does not obtain with a careful interpretation of the space of allowed perturbations.

Additionally, the results apply also to economies with nominal (rather than real) assets. Such economies are known to generate partially revealing equilibria, even without a rich structure of uncertainty (see, for example Rahi 1995).

A Farkas-Stiemke lemma for cones

In this section, all vectors are assumed to be column vectors, unless specified otherwise. Thus $x \in \mathbb{R}^n$ is regarded as a $n \times 1$ matrix, and x' denotes the transpose of x, and it is a $1 \times n$ matrix. Let $R \subset \mathbb{R}^J$ be a cone. Denote by R^* the polar cone of R:

$$R^* := \{z \in \mathbb{R}^J \mid z'\theta \geq 0, \forall \theta \in R\}.$$

Let A be an $S \times J$ real matrix, and let $p \in \mathbb{R}^J$.

Condition A.1 (adapted Slater condition)**.** *Suppose that there exists $\hat{\theta} \in R$ such that $A\hat{\theta} \in \mathbb{R}^S_{++}$.*

The following is a particular case of the main theorem in Sposito and David (1972):[12]

Lemma A.1 (Farkas lemma for cones)**.** *Suppose that Condition A.1 holds and that R is a closed convex cone. There does not exist $\theta \in R$ such that*

$$p'\theta < 0 \text{ and } A\theta \geq 0$$

if and only if there exists $\lambda \in \mathbb{R}^S_+$ such that $p - A'\lambda \in R^*$.

I prove next a strict version of Farkas' lemma for cones:

[12]The first of the two cones in the statement of their theorem is set to $\mathbb{R}^S_+$.

Proposition A.2 (Stiemke lemma for cones)**.** *Suppose that Condition A.1 holds and that R is a closed convex cone. There does not exist $\theta \in R$ such that*

$$p'\theta \leq 0 \text{ and } A\theta \geq 0, \text{ with at least one strict inequality}$$

if and only if there exists $\lambda \in \mathbb{R}^S_{++}$ such that $p - A'\lambda \in R^*$.

Proof. Sufficiency is immediate. To prove the necessity part, denote the rows of A by $a_1, \ldots, a_S$, and let $a_0 := -p'$. Let $\bar{A}$ be the $(S+1) \times J$ matrix having as rows $a_0, a_1, \ldots, a_S$. Therefore there is no $\theta \in R$ such that $\bar{A}\theta \geq 0$ and $\bar{A}\theta \neq 0$. Applying Lemma A.1 $S+1$ times, it follows that for each $s \in \{0, 1, \ldots, S\}$, there exists $\lambda^s \in \mathbb{R}^{S+1}_+$ with $\lambda^s_s = 1$ such that $\bar{A}'\lambda^s \in R^*$. Therefore $\bar{A}' \sum_{s=0}^S \lambda^s \in R^*$, and the conclusion follows, since $\sum_{s=0}^S \lambda^s \in \mathbb{R}^{S+1}_{++}$. □

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