Robust bubbles with mild penalties for default *

Florin Bidian †‡

December 17, 2013

Abstract

Limited enforcement of debt contracts and mild penalties for default can lead to low equilibrium interest rates, to ensure debt repayment. Low interest rates, in turn, create conditions for bubbles. I show that bubbles can arise as a substitute to private liquidity when the punishment for default is a permanent or a temporary interdiction to trade, and complement in this way the existing results for an interdiction to borrow as penalty for default (Hellwig and Lorenzoni 2009). The size of bubbles is jointly determined by agents' endogenous debt limits and interest rates, and is not comonotonic with the amount of risk sharing that takes place in different equilibria. Agents' endogenous debt limits are not necessarily more relaxed for agents with higher income.

Keywords: bubbles; self-enforcing debt; limited enforcement; penalties for default; endogenous debt limits; risk-sharing;

JEL classification: G12,G11,E44,D53

1 Introduction

A rational bubble is defined as the price of an asset in excess of the present value of its dividends. Santos and Woodford (1997) show that bubbles on assets in positive

---

*Some of the results in here are based on Bidian (2011, Chapter 4).

†Robinson College of Business, Georgia State University, PO Box 4036, Atlanta GA 30302-4036. E-mail: fbidian@gsu.edu

‡I thank seminar audiences at University of Minnesota, Georgia State University, Utah State University, the 2012 and 2013 Workshop on General Equilibrium (Exeter, UK and Vienna, Austria) and the 2012 North American Summer Meetings of the Econometric Society (Evanston, US) for useful comments. All remaining errors are mine.

supply cannot exist if the interest rates are high, that is, if the present value of aggregate endowment is finite.[1] Indeed, bubbles grow on average at the same rate as the interest rates and therefore they are positive martingales when discounted by the pricing kernel. With high interest rates, a bubble would become very large relative to the aggregate endowment, with positive probability. This is incompatible with the presence of optimizing, forward looking agents, who do not allow their financial wealth to be too large relative to the present value of their future consumption.

This paper argues that low interest rates arise naturally to induce repayment in economies with limited enforcement of debt contracts, and this can lead to bubbles. Agents have the option to default on debt and incur a punishment, leading to a (reduced) continuation utility that can be date and state dependent. As in Alvarez and Jermann (2000), markets select endogenously the largest debt limits for the agents so that repayment is always individually rational given future bounds on debt.

When the punishment for default is an *interdiction to borrow (IB)*, Hellwig and Lorenzoni (2009) show that any non-autarchic equilibrium must have low interest rates, and agents' endogenous discounted (by the pricing kernel) debt limits are martingales. Any such equilibrium with private debt is equivalent to an equilibrium where no borrowing and lending between agents (private debt) is allowed, but in which agents can acquire government debt. The public debt is "unbacked" and simply rolled over. In a deterministic economy, this unbacked public debt is equivalent to a fixed supply of fiat money, and therefore can be viewed as a bubble.

They establish the existence of non-autarchic equilibria in a Markov economy with two agents with endowments switching with constant probability between a high value and a low value, but with constant aggregate endowment. There is a stationary equilibrium and a continuum of nonstationary equilibria converging to autarchy. All equilibria have low interest rates and can sustain bubbles. The size of (initial) bubbles comoves with (is in fact equal to) the (initial) amount of risk-sharing (transfers) between agents.

In the same environment, I document the existence of bubbles for the other most common penalties for default encountered in the literature: a permanent *interdiction to trade (IT)* after default (Kehoe and Levine 1993, 2001, or Alvarez and Jermann

[1] For deterministic economies, the results of Santos and Woodford (1997) were anticipated by Kocherlakota (1992) and later refined by Huang and Werner (2000).

2000, 2001), a temporary *interdiction to trade for a finite (ITF)* (and deterministic) number of periods (Azariadis and Kaas 2008), or an *interdiction to trade for a random (ITR)* number of periods (Azariadis and Kaas 2013). Under penalty (IT), a defaulting agent is permanently excluded from the markets and consumes his endowment. With penalty (ITF), after default an agent consume his endowment for a predetermined number of periods, after which full trading privileges are restored. Finally, under penalty (ITR), an agent in default permanently regains full access to the markets with some fixed probability per period.

Penalty (IT) is more severe than (IB),(ITF) and (ITR), and is, in a sense, the harshest penalty that can still sustain low interest rates. Indeed, if creditors can confiscate an arbitrarily small fraction of the current and future income of a defaulting agent (in addition to banning him from the markets), Bloise, Reichlin, and Tirelli (2013, Appendix B) show that high interest rates must arise in equilibrium. The reason is that each agent's debt limits, in absolute value, are bounded from below by the present value of the fraction of agent's endowment that is garnished upon default.

Finding equilibria with low interest rates does not guarantee the existence of bubbles. For example, under all the punishments for default and parameters considered in this paper, autarchy is an equilibrium with low interest rates, but agents' debt limits are zero and no bubbles can be sustained. The necessary and sufficient condition for the sustainability of bubbles is the existence of martingale components in agents' discounted debt limits, which is a joint property of the endogenously determined interest rates and debt bounds. Bubbles equal to the total martingale components in agents' debt limits are possible.

I show, therefore, that agents' discounted debt limits have martingale components. These components can be converted into bubbles (valued fiat money) using Kocherlakota's (2008) bubble equivalence theorem - any bubble-free equilibrium is equivalent to a bubbly equilibrium in which agents' debt limits are tightened by the bubble times their initial endowment of the asset. In contrast to Hellwig and Lorenzoni (2009)), under the penalties (IT), (ITF), (ITR) considered here (and in general, for any penalties other than (IB)), private debt cannot be substituted entirely by unbacked public debt (bubbles), as the discounted agents' debt limits are not martingales.[2] Therefore bubbles (unbacked public debt) must coexist with private debt.

I consider first the penalty (IT). In addition to a *stationary* equilibrium with high interest rates, there is a continuum of *nonstationary* equilibria with low interest rates that can sustain bubbles. The multiplicity of equilibria was recognized also by Bloise, Reichlin, and Tirelli (2013) and Antinolfi, Azariadis, and Bullard (2007). However, they have not calculated the endogenous debt limits and show the existence of martingale components in them.

For stationary equilibria associated to penalty (IT), Alvarez and Jermann (2001) and Ábrahám and Cárceles-Poveda (2010) found that agents with higher income have more relaxed debt limits. In other words, the level of debt that makes agents indifferent between defaulting or not is higher for higher income agents. However, this finding is not robust and does not apply to the nonstationary equilibria. There are two opposite forces at work. A higher level of income makes default more attractive, and therefore the value of the outside option is larger. However, higher income agents might value more the access to markets and the possibility to save given binding debt constraints in the future. In the nonstationary equilibria the first effect can dominate, and high-income agents can have tighter debt limits at some dates.[3] Moreover, in contrast to the penalty (IB) (see Hellwig and Lorenzoni 2009), the size of sustainable bubbles is not comonotonic with the amount of risk-sharing (transfers between agents) arising in the different equilibria.[4]

For the (ITF) penalty, I consider first the deterministic case and show the existence of a non-autarchic stationary equilibrium that can sustain bubbles, if interdiction to trade is sufficiently short-lived.[5] This equilibrium allows for less risk sharing and smaller bubbles than in the stationary equilibrium under penalty (IB). Penalty (ITF) can also lead to debt limits that are tighter for high income agents tghan for low income agents at all dates (Section 5).

[2]In fact, discounted debt limits are submartingales for penalty (IT) (Bidian and Bejan 2012) and can be supermartingales for (ITF), as seen in Section 5.

[3]In these equilibria the actual allocation of debt limits between agents is indeterminate, and is achieved by varying agents' initial wealth (or their starting wealth after a transition phase initiated from fixed levels of wealth). As seen in Proposition 2.1, martingale components added to discounted debt limits leave agents' budget constraints unchanged if the initial wealth of the agent is increased by the initial value of the martingale.

[4]It will be shown that the size of the sustainable bubble is not equal anymore to the current amount of transfers, but rather to the expected discounted value of asymptotic transfers.

[5]Moreover, these equilibria are robust to the introduction of small bankruptcy costs, in the sense of Azariadis and Kaas (2013).

I show then the (generic) existence of stationary stochastic equilibria for penalty (ITR), when the probabilities of reentering the market and of a state reversal are high. These equilibria have low interest rates and can sustain bubbles. The stochastic case for penalty (ITF) with a one-period interdiction to trade after default is a particular case of the penalty (ITR) when agents reenter the markets with probability one upon default.

The results of this paper suggests that low interest rates and rational bubbles are the norm (rather the exception) in economies with limited enforcement and mild penalties for default. This is in sharp contrast with the view that rational bubbles are fragile/special. For example, Montrucchio and Privileggi (2001) show, under weak assumptions on agent's preferences, that bubbles cannot exist in a representative agent economy. As mentioned earlier, even with heterogeneous agents, Santos and Woodford (1997) prove that bubbles are only possible if interest rates are low.

Low interest rates are seen as unplausible, as they are ruled out by the presence of an asset with dividends higher that some fraction of the aggregate endowment. Abel, Mankiw, Summers, and Zeckhauser (1989), testing the dynamic efficiency of US and six other developed economies, found that cash flows generated by the capital sector exceed uniformly the level of investment. This finding was interpreted as evidence of high interest rates. Geerolf (2013), however, using an updated data set and a different measurement of land rents, overturns the conclusion of Abel, Mankiw, Summers, and Zeckhauser (1989) and finds evidence of dynamic inefficiency and low interest rates.

A discussion of the challenges faced by the empirical literature on low interest rates and rational bubbles in international debt is provided in Hellwig and Lorenzoni (2009). Bidian (2011) also discusses the inherent difficulties and pitfalls in testing for rational bubbles in stock prices. The contradictory findings of the empirical literature on bubbles underscores the importance of a better (theoretical) understanding of the conditions and frictions that can give rise to bubbles. The paper is part of this mission.

2 Model

2.1 Setup

Time periods are indexed by $\mathbb{N} := \{0, 1, \ldots\}$. The uncertainty is described by a time homogeneous Markov process $(s_t)_{t\in\mathbb{N}}$ with states $\{1, 2\}$, and with a probability of reversal equal to $\pi \in (0, 1]$. Thus for any $t \in \mathbb{N}$, $s_t \in \{1, 2\}$ and $s_{t+1} \neq s_t$ with probability π. Let X be the set of stochastic processes adapted to the information generated by $(s_t)_{t\in\mathbb{N}}$. Thus $x = (x_t)_{t\in\mathbb{N}} \in X$ if for each t, $x_t : \{1, 2\}^{\mathbb{N}} \to \mathbb{R}$, and $x_t(s_0, s_1, \ldots)$ depends only on $s_0, s_1, \ldots, s_t$. Denote by X_+ (X_{++}) the set of non-negative (strictly positive) processes in X. The conditional expectation given the information available at t is denoted by $E_t(\cdot)$.

There are two agents $I := \{1, 2\}$ with identical utilities on consumption processes $c \in X$ given by $E_0 \sum_{t\geq 0} \beta^t u(c_t)$, where u is strictly increasing, strictly concave and continuously differentiable, and $\beta \in (0, 1)$. At each date t, agent i receives an income $y^i_t := y^H$ if $s_t = i$ and $y^i_t := y^L$ otherwise, with $y^H > y^L$. The agent with income y^H at t is referred to as the *high-type* at t, and the agent with income y^L at t is the *low-type* at t.

At each date t agents can trade in a complete set of one period Arrow securities. The price at t of the Arrow security paying one unit of consumption at $t + 1$ if the state changes (that is, $s_{t+1} \neq s_t$), respectively does not change ($s_{t+1} = s_t$), is πq^c_t, respectively $(1 - \pi)q^{nc}_t$. The pricing kernel $p \in X_{++}$ is defined by $p_0 := 1$ and $\frac{p_{t+1}}{p_t}$ equals q^c_t if $s_{t+1} \neq s_t$ and q^{nc}_t otherwise. There is also an infinitely-lived asset trading at prices $b \in X_+$, paying zero dividends and assumed in unit supply. This asset is referred to as (fiat) *money*. Each agent $i \in I$ has an initial nonnegative endowment of money $\theta^i_{-1} \geq 0$, and additional wealth (in the form of Arrow securities) a^i_0. He faces debt constraints requiring his beginning of period financial wealth to exceed some negative bounds $\phi^i \in -X$, meant to prevent Ponzi schemes. Consider an agent i that starts period t with holdings a_t of Arrow securities and θ_{t-1} of money, thus with a financial wealth $\nu_t := a_t + b_t\theta_{t-1}$. Facing constraints ϕ and prices p, b, he maximizes his *continuation utility* $U^i_t(c)$ subject to budget constraints,

$$\max_{(c,a,\theta)\in B^i_t(\nu_t,\phi^i,p,b)} U^i_t(c) := E_t \sum_{s\geq t} \beta^{s-t} u(c_s),$$

where $B_t^i(\nu_t, \phi^i, p, b)$ is his budget constraint following t, defined as

$$
\begin{aligned}
B_t^i(\nu_t, \phi^i, p, b) := \{ &(c_s, a_{s+1}, \theta_s)_{s \geq t} \,|a_s + b_s \theta_{s-1} \geq \phi_s^i, \\
&c_s + E_s \frac{p_{s+1}}{p_s} a_{s+1} + p_s \theta_s \leq y_s^i + a_s + b_s \theta_{s-1}, \forall s \geq t \}.
\end{aligned} \tag{2.1}
$$

The optimal continuation utility (indirect utility) of the agent is given by

$$
V_t^i(\nu_t, \phi^i, p, b) := \max_{(c,a,\theta) \in B_t^i(\nu_t, \phi^i, p, b)} U_t^i(c). \tag{2.2}
$$

When there is no ambiguity about the future debt limits and prices faced by agent i, I will denote $V_t^i(\nu_t, \phi^i, p, b)$ simply by $V_t^i(\nu_t)$. The notation $V_t^H(\nu_t)$, respectively $V_t^L(\nu_t)$ refers to the indirect utility of the high-type agent, respectively low-type agent starting period t with wealth ν_t.

2.2 Equilibrium with endogenous debt limits

Consumer i can elect to default on his debt at any period t and receive a "penalty" continuation utility $V_t^{i,d}(p, b, \phi^i)$ that can depend on exogenous variables such as endowments, but also on prices p, b, and even future debt limits $\phi_{t+1}^i, \phi_{t+2}^i, \ldots$. When no confusion can arise, I use simply $V_t^{i,d}$ to denote agent's i continuation utility if he defaults at t.

The debt limits ϕ^i are determined endogenously to reflect the maximal amount of debt agents can hold without defaulting. Debt limits ϕ^i are *self-enforcing* for agent i at prices p, b given penalties $V^{i,d}$ if $B_t^i(\phi_t^i, \phi^i, p, b) \neq \emptyset$ for all $t \in \mathbb{N}$ and the agent prefers *not* to default, $V_t^i(\phi_t^i, \phi^i, p, b) \geq V_t^{i,d}, \forall t \in \mathbb{N}$. The debt limits ϕ^i are *not-too-tight* (Alvarez and Jermann 2000) for agent i (at prices p, b) given penalties $V^{i,d}$ if and only if

$$
V_t^i(\phi_t^i, \phi^i, p, b) = V_t^{i,d}, \forall t \in \mathbb{N}. \tag{NTT}
$$

Thus not-too-tight debt limits are self-enforcing bounds that do not restrict credit unnecessarily.

Alvarez and Jermann (2000), building on the work of Kehoe and Levine (1993), assume that the agents are subject to a permanent *interdiction to trade* following default, that is

$$
V_t^{i,d} := U_t^i(y^i), \forall t \in \mathbb{N}. \tag{IT}
$$

Hellwig and Lorenzoni (2009), following Bulow and Rogoff (1989), assume that agents face an *interdiction to borrow* upon default. Hence agents can renege on their debt and be required to hold nonnegative wealth thereafter, resulting in a continuation utility that depends on prices,

$$V_t^{i,d} := V_t^i(0,0,p,b), \forall t \in \mathbb{N}, \tag{IB}$$

where the second argument in $V_t^i(0,0,p,b)$ denotes the zero debt limits (the zero process). Alternatively, agents can be subject to an *interdiction to trade for a finite* number of periods $M \geq 1$, and their holdings of money cannot be confiscated at default time:

$$V_t^{i,d} := E_t \left(\sum_{s=0}^{M-1} \beta^s u(y_{t+s}^i) + \beta^M V_{t+M}^i(b_{t+M}\theta_{t-1}^i, \phi^i, p, b) \right), \forall t \in \mathbb{N}. \tag{ITF}$$

Azariadis and Kaas (2013) analyzed a variant of (ITF), and assumed that agents face an *interdiction to trade for a random* number of periods. When a consumer defaults, he cannot trade securities in the default period. His holdings of money cannot be confiscated. In any subsequent period, the consumer regains full access to the markets with probability $\mu \in [0,1]$. In this case,

$$V_t^{i,d} := E_t \sum_{s \geq 0} (1-\mu)^s \beta^s \left(u(y_{t+s}^i) + \mu \beta E_t V_{t+s}(b_{t+s}\theta_{t-1}^i, \phi, p, b) \right). \tag{ITR}$$

A vector $\left(p, b, (c^i)_{i\in I}, (a^i)_{i\in I}, (\theta^i)_{i\in I}, (\phi^i)_{i\in I}, (V^{i,d})_{i\in I}\right)$ consisting of prices p, b, consumption (c^i), asset holdings $(a^i), (\theta^i)$, debt constraints (ϕ^i) and penalties for default $(V^{i,d})$ is an *equilibrium* with initial money holdings $(\theta_{-1}^i)_{i\in I}$ and initial holdings of wealth in the form of Arrow securities $(a_0^i)_{i\in I}$ if

i. Consumption and portfolios of each agent i are feasible and optimal: $(c^i, a^i, \theta^i) \in B_0^i(a_0^i + b_0\theta_{-1}^i, \phi^i, p, b)$ and $U(c^i) = V_0^i(a_0^i + b_0\theta_{-1}^i, \phi^i, p, b)$.

ii. Markets clear: $\sum_{i\in I} c_t^i = \sum_{i\in I} y_t^i$, $\sum_{i\in I} \theta_{t-1}^i = 1$, $\sum_{i\in I} a_t^i = 0, \forall t \geq 0$.

iii. For each $i \in I$, ϕ^i is not-too-tight given $V^{i,d}$: $V_t^i(\phi_t^i, \phi^i, p, b) = V_t^{i,d}$, $\forall t \geq 0$.

2.3 Rational bubbles

Money is a redundant security and plays no role in this economy. As money pays no dividends, valued money will be referred to as a *bubble*. The absence of arbitrage opportunities in an equilibrium implies that

$$p_t b_t = E_t p_{t+1} b_{t+1}, \forall t \geq 0. \tag{2.3}$$

The value of money (b_t) has to grow on average at the rate of interest rates, or equivalently, the discounted value of money is a martingale. This is the intrinsic property of a rational bubble: *discounted bubbles* are martingales. As a consequence, there exists a t such that $b_t \neq 0$ at some t if and only the initial value of money is nonzero, $b_0 \neq 0$.

Due to the redundancy of money, any equilibrium is equivalent with an equilibrium in which agents keep their holdings of money fixed to the initial level and trade only in the Arrow securities. In other words, if $(p, b, (c^i), (a^i), (\theta^i), (\phi^i), (V^{i,d}))$ is an equilibrium, then $(p, b, (c^i), (\bar{a}^i), (\bar{\theta}^i), (\phi^i), (V^{i,d}))$ is an equilibrium with *no trading in money*, that is with $\bar{\theta}^i_{t-1} = \theta^i_{-1}$, and in which $\bar{a}^i_{t+1} = a^i_{t+1} + b_{t+1}(\theta^i_t - \theta^i_{-1})$ for all $t \geq 0$ and all $i \in I$. We focus throughout only on equilibria with no trading in money.

For the rest of the paper, the penalty for default is assumed to be (IT), (IB),(ITF) or (ITR). With these penalties, it is also sufficient to focus only on equilibria with unvalued money. Any equilibrium having a bubble (valued money) is equivalent with a bubble-free equilibrium in which an agent's debt limits are relaxed by the size of the bubble times his (initial) holdings of money. Conversely, given an equilibrium with unvalued money in which the endogenous discounted debt limits have martingale components (are lower than a negative martingale) is equivalent with a bubbly equilibrium in which the martingale components in debt limits are converted into valued money, creating a bubble. These claims follow by extending Kocherlakota's (2008) *bubble equivalence theorem* to the more general penalties for default considered here (that can depend on the endogenous debt limits).

Proposition 2.1. *(a) Let $(p, b, (c^i), (a^i), (\theta^i), (\phi^i), (V^{i,d}))$ be an equilibrium (with valued money). Then $(p, 0, (c^i), (a^i), (\theta^i), (\bar{\phi}^i), (V^{i,d}))$ is an equilibrium, where $\bar{\phi}^i_t = \phi^i_t - b_t\theta^i_{-1}$, for all $t \geq 0$ and $i \in I$.*

(b) Let $(p, 0, (c^i), (\bar{a}^i), (\bar{\theta}^i), (\bar{\phi}^i), (V^{i,d}))$ be an equilibrium (with unvalued money).

For each $i \in I$, let $\varepsilon^i \in X_+$ such that $p \cdot \varepsilon^i$ is a martingale and $\phi^i \leq -\varepsilon^i$. Then $(p, b, (c^i), (a^i), (\theta^i), (\phi^i), (V^{i,d}))$ is an equilibrium with $b = \sum_{i=1}^{I} \varepsilon^i$ and

$$\phi^i = \bar{\phi}^i + \varepsilon^i, \quad a^i = \bar{a}^i + \varepsilon^i, \quad \theta^i = 0, \text{ for } i \neq 1,$$
$$\phi^1 = \bar{\phi}^1 + \varepsilon^1, \quad a^1 = \bar{a}^1 + \varepsilon^1 - \sum_{i \in I} \varepsilon^i, \quad \theta^1 = 1.$$

Proof. (a) Since $p(\bar{\phi}^i - \phi^i) = -pb\theta^i_{-1}$ is a martingale,

$$(c, a, \theta) \in B^i_t(\nu_t, \phi^i, p, b) \Leftrightarrow (c, a + \bar{\phi}^i - \phi^i, \theta) \in B^i_t(\nu_t + \bar{\phi}^i_t - \phi^i_t, \bar{\phi}^i, p, 0), \tag{2.4}$$

and therefore

$$V^i_t(\nu_t, \phi^i, p, b) = V^i_t(\nu_t + \bar{\phi}^i_t - \phi^i_t, \bar{\phi}^i, p, 0). \tag{2.5}$$

Market clearing conditions hold. All that is left is to show that the tighter debt limits $\bar{\phi}^i$ are not-too-tight, that is $V^i_t(\bar{\phi}^i_t, \bar{\phi}^i, p, 0) = V^{i,d}(p, 0, \bar{\phi}^i)$. Setting $\nu_t := \phi_t$ in (2.5), $V^i_t(\bar{\phi}^i_t, \bar{\phi}, p, 0) = V^i_t(\phi_t, \phi, p, b)$. Since ϕ^i are not-too-tight, $V^i_t(\phi_t, \phi, p, b) = V^{i,d}(p, b, \phi^i)$. Therefore $\bar{\phi}^i$ are not-too-tight if and only if

$$V^{i,d}(p, 0, \bar{\phi}^i) = V^{i,d}(p, b, \phi^i). \tag{2.6}$$

For penalties (IT) and (IB), (2.6) holds as agent's i continuation utilities after default do not depend on the debt limits $\phi^i, \bar{\phi}^i$, and depend on b only through the pricing kernel p, by the absence of arbitrage opportunities (2.3). For punishments (ITF),

$$\begin{aligned} V^{i,d}_t(p, b, \phi^i) &= E_t\left(\sum_{s=0}^{M-1} \beta^s u(y^i_{t+s}) + \beta^M V^i_{t+M}(b_{t+M}\theta^i_{t-1}, \phi, p, b)\right) \\ &= E_t\left(\sum_{s=0}^{M-1} \beta^s u(y^i_{t+s}) + \beta^M V^i_{t+M}(0, \bar{\phi}, p, 0)\right) = V^{i,d}_t(p, 0, \bar{\phi}^i), \end{aligned} \tag{2.7}$$

where the first and last equality follow from the definition of penalties (ITF), while the middle equality holds by (2.5). An identical reasoning shows that (2.6) holds for penalty (ITR).

(b) Market clearing conditions hold. Agents' optimality conditions are satisfied, by (2.4). The not-too-tight property can be verified as above (see (2.5)-(2.7)). $\square$

2.4 Constrained equilibria

Proposition 2.1 shows that the existence of bubbles (valued money) (bubbles) is tantamount to the existence of martingale components in agents' (discounted) debt limits arising in an equilibrium without money. We focus henceforth on equilibria $(p, (c^i), (a^i), (\phi^i), (V^{i,d}))$ with *unvalued* money ($b = 0$) and analyze the existence of such martingale components. The first order conditions for agent $i \in I$ (at allocations with positive consumption) are

$$\frac{p_{t+1}}{p_t} \geq \frac{\beta u'(c^i_{t+1})}{u'(c^i_t)}, \text{ with “=” if } a^i_{t+1} > \phi^i_{t+1}. \tag{2.8}$$

I assume that there is enough heterogeneity in agents' income or that the discount rate β and the probability π of reversals are high enough so that

$$\tilde{\beta}\frac{u'(y^L)}{u'(y^H)} > 1, \text{ where } \tilde{\beta} := \frac{\beta\pi}{1-\beta+\beta\pi}. \tag{2.9}$$

I analyze the equilibria where the debt limit is binding in the high-endowment state for each agent. Thus high-type agents always start the period with wealth equal to their debt limit, and are indifferent between defaulting or not. I will refer to such equilibria as *constrained*. Moreover, at any date n, the Arrow security prices and agents' consumption, asset holdings and debt limits are the *same* in all histories $(s_0, s_1, \ldots, s_n)$ having the same *number of state reversals*

$$\tau_n := \sum_{k=1}^{n} |s_k - s_{k-1}|. \tag{2.10}$$

Denote by $x_t \in [0, y^H)$ the *transfer* from the high-type agent to the low-type agent at histories with t reversals. In other words, consumption of the high-type (respectively low-type) at some date n is $y^H - x_{\tau_n}$ (respectively $y^L + x_{\tau_n}$). Similarly, let $(1-\pi)q^{nc}_t$ and πq^c_t be the Arrow security prices in histories with t reversals. The first order conditions (2.8) for the two types of agents amount to (with

$$q_t^{nc} = \beta, \quad \frac{\beta u'(y^L + x_{t+1})}{u'(y^H - x_t)} = q_t^c, \tag{2.11}$$

$$\frac{\beta u'(y^H - x_{t+1})}{u'(y^L + x_t)} \leq q_t^c, \forall t \geq 0. \tag{2.12}$$

Using (2.11), inequality (2.12) can be written as

$$\frac{u'(y^L + x_t)}{u'(y^H - x_t)} \geq \frac{u'(y^L + (y^H - y^L - x_{t+1}))}{u'(y^H - (y^H - y^L - x_{t+1}))},$$

which holds if and only if

$$x_t + x_{t+1} \leq y^H - y^L. \tag{2.13}$$

Therefore transfers (x_t) and Arrow security prices (q_t^c, q_t^{nc}) are compatible with agents' first order conditions if and only if (2.11) and (2.13) are satisfied. Denote by a_t the beginning of period wealth of the low-type agent in all histories with t reversals. In a constrained equilibrium, agents' budget constraints are

$$x_t - \pi q_t^c a_{t+1} + (1 - \pi) q_t^{nc} a_t = a_t. \tag{2.14}$$

The debt limit ϕ_t^H of a high-type agent in histories with t reversals is

$$\phi_t^H = -a_t. \tag{2.15}$$

Constrained equilibria are thus entirely determined by the sequences of transfers and beginning of period asset holdings and debt limits of low-types $((x_t), (a_t), (\phi_t^L))$.

Consider a constrained equilibrium $((x_t), (a_t), (\phi_t^L))$. What happens if the asset holdings of the initial low-type are different from a_0 and equal to some a^0? In this case there is a *transition* phase that lasts until the first state reversal, and then the economy enters the paths described by the given constrained equilibrium. During the transition, transfers and asset holdings are constant, and equal to x^0 and a^0. Let be the transfers and asset holdings of low-types during the transition. The price of the Arrow security contingent on a reversal next period ending the transition is πq^0, while the price of the Arrow security contingent on no reversal is $(1 - \pi)\beta$. Then x^0, a^0 followed by transfers and asset holdings $(x_t), (a_t)$ (after the transition ends) form an equilibrium if agents' participation constraints in the initial period are satisfied (initial utilities along the equilibrium path exceed the penalty continuation utilities) and if agents' budgets and the (necessary and sufficient) first order conditions hold,

$$x^0 = \pi q^0 a_0 - (1-\pi)\beta a^0 + a^0,\ q^0 = \beta \frac{u'(y^L + x_0)}{u'(y^H - x^0)},\ x^0 + x_0 \leq y^H - y^L. \tag{2.16}$$

Assume, for concreteness, that both agents start with zero wealth, thus $a^0 = 0$. Under all penalties considered here, the optimal continuation utilities of an agent with zero initial wealth exceed the penalty continuation utilities (following default), as the optimal paths following default are budget feasible for a non-defaulting agent starting with zero wealth. By the equalities in (2.16), x^0 is determined uniquely by

$$x^0 u'(y^H - x^0) = \beta\pi a_0 u'(y^L + x_0), \tag{2.17}$$

as the left hand side in (2.17) is strictly increasing. Thus x^0, a^0 followed by $(x_t), (a_t)$ after the transition constitute an equilibrium if and only if $x^0 + x_0 \leq y^H - y^L$, or equivalently, by (2.17), if and only if

$$a_0 \leq (\beta\pi)^{-1}(y^H - y^L - x_0). \tag{2.18}$$

When the penalty for default is the *interdiction to borrow* (IB), Hellwig and Lorenzoni (2009) show that there exists a unique (from the point of view of agents' consumption) constrained stationary equilibrium, characterized by transfers $x_t = x^*$ for all t such that equilibrium interest rates are zero, $\pi q_t^c + (1-\pi) q_t^{nc} = 1$, or alternatively,

$$u'(y^H - x^*) = \tilde{\beta} u'(y^L + x^*), \tag{2.19}$$

with $\tilde{\beta}$ defined in (2.9). For any initial transfer $0 < x_0 < x^*$, there are nonstationary equilibria (x_t) converging monotonically to autarchy, $x_t \searrow 0$. In the stationary and nonstationary equilibria, agents' discounted debt limits are *martingales* and $x_t = -\sum_{i \in I} \phi_t^i$. All these equilibria can sustain bubbles of initial size x_0 (Proposition 2.1). If there is no uncertainty and the state alternates deterministically between the states $1, 2$ (that is, if $\pi = 1$), then discounted debt limits are constant. Therefore debt limits are $\phi_t^1 = -\lambda x_t$, $\phi_t^2 = -(1-\lambda)x_t$, for all t, where $\lambda \in [0,1]$ is arbitrary (tighter debt limits for agent 1 are compensated by higher initial wealth and viceversa).

In the analysis of the other penalties for default ((IT), (ITF), (ITR)), I analyze first the *deterministic* case $\pi = 1$. The general stochastic case ($\pi < 1$) builds on the deterministic case. Notice that for $\pi = 1$, $\tilde{\beta} = \beta$. The bond prices (q_t^c) are denoted simply by (q_t). Given transfers (x_t), bond prices (q_t) are determined by (2.11), while the pricing kernel is

$$p_0 = 1, p_{t+1} = q_0 q_1 \ldots q_t, \forall t \geq 0. \tag{2.20}$$

Asset holdings are obtained by iterating in (2.14),

$$(-1)^{t+1} a_{t+1} = \frac{a_0 - \sum_{s=0}^{t} (-1)^s p_s x_s}{p_{t+1}}. \tag{2.21}$$

3 Permanent interdiction to trade

Alvarez and Jermann (2001) analyzed only the stationary constrained equilibria associated to this penalty. Antinolfi, Azariadis, and Bullard (2007) and Bloise, Reichlin, and Tirelli (2013, Section 2) focused on the deterministic case and pointed out that, in addition to the stationary constrained equilibrium, there are an infinite number of nonstationary ones converging to autarchy. However, they have not computed the not-too-tight debt limits supporting these allocations and analyzed the asymptotic behavior of discounted debt limits, which is crucial for understanding whether bubbles can be sustained in equilibrium.

3.1 The deterministic case

Consider a constrained equilibrium $((x_t), (a_t), (\phi_t^L))$. As high-types are indifferent between defaulting or not,

$$0 = f(x_t, x_{t+1}) := u(y^H) + \beta u(y^L) - u(y^H - x_t) - \beta u(y^L + x_{t+1}). \tag{3.1}$$

Knowing the transfers x_t at t, (3.1) gives the transfers x_{t+1} (at $t+1$) as a function of x_t, $x_{t+1} = x^{next}(x_t)$. Let $\bar{x}, \bar{x}' \in (0, y^H - y^L)$ satisfying $f(\bar{x}, \bar{x}) = 0$ and $f(\bar{x}', y^H -$

$y^L - \bar{x}') = 0$, that is

$$u(y^H - \bar{x}) + \beta u(y^L + \bar{x}) = u(y^H) + \beta u(y^L), \tag{3.2}$$

$$(1+\beta)u(y^H - \bar{x}') = u(y^H) + \beta u(y^L). \tag{3.3}$$

Proposition 3.1 establishes the existence and uniqueness of $\bar{x}, \bar{x}'$ and shows that either $\bar{x} \geq \bar{x}' \geq \frac{y^H - y^L}{2}$ or $\bar{x} < \bar{x}' < \frac{y^H - y^L}{2}$. By construction, $\bar{x} = x^{next}(\bar{x})$, thus $\bar{x}$ represents the (non-zero) stationary solution of the difference equation (3.1). On the other hand, $\bar{x}' + x^{next}(\bar{x}') = y^H - y^L$, hence at transfer levels higher than $\bar{x}'$, low-types become unconstrained and (2.13) is violated.

Proposition 3.1. *Choose x_0 such that $0 \leq \min\{\bar{x}, \bar{x}'\}$. There exists a unique sequence $(x_t)_{t\geq 0}$ satisfying $f(x_t, x_{t+1}) = 0$ for all $t \geq 0$, and $(x_t)_{t\geq 0}$ is strictly decreasing to 0 if $x_0 \notin \{0, \bar{x}\}$ and constant if $x_0 \in \{0, \bar{x}\}$. Moreover, (x_t) are the transfers in a constrained equilibrium with asset holdings (2.21), where initial asset holdings a_0 can be arbitrarily chosen in $[\underline{L}, \bar{L}]$, with $\underline{L} := \lim_{t\to\infty} L_{2t-1}$, $\bar{L} := \lim_{t\to\infty} L_{2t} = \underline{L} + \lim_{t\to\infty} p_t x_t$ and $L_t := \sum_{s=0}^{t} (-1)^s p_s x_s$. Agents' discounted debt bounds $(p_t \phi_t^1), (p_t \phi_t^2)$ are increasing sequences, and $\phi_t^L = -x_t - \phi_t^H = \frac{p_{t+1}}{p_t} \phi_{t+1}^H$, for all $t \geq 0$.*

The proof is given in Appendix A. Since agents' discounted debt limits are increasing sequences, and the total credit in the economy equals the equilibrium transfers, $x_t = -\sum_{i\in I} \phi_t^i$, it follows that a bubble of maximal initial size

$$-\lim_{t\to\infty} p_t(\phi_t^e + \phi_t^o) = \lim_{t\to\infty} p_t x_t \tag{3.4}$$

can be sustained in equilibrium (Proposition (2.1)).

Proposition 3.1 shows that $\bar{x}$ defined in (2.19) represents the transfers in a *stationary* constrained equilibrium if $\bar{x} \leq (y^H - y^L)/2$, or equivalently, if[6]

$$(1+\beta)u((y^H + y^L)/2) \leq u(y^H) + \beta u(y^L). \tag{3.5}$$

Bond prices are constant in this stationary equilibrium and equal to $q(\bar{x}) := \beta u'(y^L + \bar{x})/u'(y^H - \bar{x}) < 1$ as $\bar{x} > x^*$ (with x^* given in (2.19)), as shown at the beginning

[6]The inequality (3.5) can be understood as requiring that the first best symmetric allocation in which each agent consumes half of the aggregate endowment does not satisfy the participation constraints of the high type agents.

of the proof of Proposition 3.1. Therefore $\lim p_t \bar{x} = \lim q^t(\bar{x})\bar{x} = 0$. No bubbles are possible, since (3.4) does not hold, or alternatively, because interest rates are high. Debt limits are

$$\phi^H = -\frac{\bar{x}}{1+q(\bar{x})}, \quad \phi^L = -\frac{q(\bar{x})\bar{x}}{1+q(\bar{x})}. \tag{3.6}$$

If (3.5) is violated, there exists an unconstrained *stationary* equilibrium with perfect risk-sharing between agents, that is with transfers $(y^H - y^L)/2$. Again interest rates are high as bond prices are $q_t = \beta$, and no bubbles are possibles. Debt limits are[7]

$$\phi^H = \frac{1}{1-\beta}\left(y^H - \bar{x}' - \frac{y^H + \beta y^L}{1+\beta}\right), \ \phi^L = \beta \phi_t^H.$$

If, on the other hand, $\bar{x} \geq (y^H - y^L)/2$, that is if (3.5) is violated, then the constant sequence of transfers equal to $(y^H - y^L)/2$ (in which consumers get half of the aggregate endowment) leads to equilibrium bond prices equal to $q = \beta < 1$ (high-interest rates), and debt limits given by (3.6), with $\bar{x}$ replaced by $(y^H - y^L)/2$ and $q(\bar{x})$ replaced by β.

However, in all the *nonstationary* equilibria constructed in Proposition 3.1, the actual allocation of debt limits between agents is indeterminate. Indeed, it will be shown next that $\lim p_t x_t > 0$ and therefore $\underline{L} < \bar{L}$. The indeterminacy in debt limits is achieved by varying agents' initial wealth. As seen in Proposition 2.1, martingales added to discounted debt limits leave agents' budget constraints unchanged if the initial wealth of the agent is increased by the initial value of the martingale.

Proposition 3.2. *Assume that agents have hyperbolic absolute risk aversion (HARA) utility functions. Any nonstationary constrained equilibrium associated to transfers* (x_t) *with* $0 < x_0 < \min\{\bar{x}, \bar{x}'\}$ *and* $f(x_t, x_{t+1}) = 0$ *for all* $t \geq 0$ *(as described in Proposition 3.1) satisfies* $\lim_{t\to\infty} p_t x_t > 0$*, and therefore can sustain bubble injections, by* (3.4).

Proof. Concavity of u implies that

$$u'(y^H)x_t < u(y^H) - u(y^H - x_t) = \beta\left(u(y^L + x_{t+1}) - u(y^L)\right) < \beta u'(y^L)x_{t+1}.$$

[7]Low-types starting with wealth $\phi^L < 0$ will borrow the maximum amount, hence $u(y^L + \phi^L - \beta\phi^H) + \beta V^H(-a) = V^{L,d} = u(y^L) + \beta V^H(-a)$, which implies $\phi^L = \beta\phi^H$. A high-type starting with wealth ϕ^H and facing bond prices β will consume a constant amount c at all dates, which implies $\frac{u(c)}{1-\beta} = \frac{u(y^H)+\beta u(y^L)}{1-\beta^2}$.

Therefore $x_{t+1}/x_t > u'(y^H)/(\beta u'(y^L)$, and by (2.11),

$$p_{t+1}x_{t+1} = p_0 x_0 \prod_{s=0}^{t} \frac{x_{s+1}/x_s}{p_s/p_{s+1}} \geq x_0 \prod_{s=0}^{t} \frac{u'(y^H)/u'(y^H - x_s)}{u'(y^L)/u'(y^L + x_{s+1})}.$$

Agents have HARA utilities $u(c) := (\alpha + \gamma c)^{1-\frac{1}{\gamma}} / (\gamma - 1)$ defined on $\{c \mid -\alpha < \gamma c\}$ (Leroy and Werner 2001, p.96), with $\alpha, \gamma \geq 0$ and $\alpha + \gamma > 0$ (so that any positive consumption belongs to the allowed domain). As usual, for $\gamma = 1$, $u(c) := \ln(\alpha + c)$ and for $\gamma = 0$, $u(c) := -e^{-\alpha c}$.

For $\gamma > 0$ (that is, for power or log utilities),

$$\frac{u'(y^H)/u'(y^H - x_s)}{u'(y^L)/u'(y^L + x_{s+1})} = \left(\frac{\left(\alpha + \gamma(y^H - x_s)\right) / \left(\alpha + \gamma y^H\right)}{\left(\alpha + \gamma(y^L + x_{s+1})\right) / \left(\alpha + \gamma y^L\right)} \right)^{\gamma} = \left(\frac{1 - \gamma x_s / (\alpha + \gamma y^H)}{1 + \gamma x_{s+1} / (\alpha + \gamma y^L)} \right)^{\gamma}.$$

As $x_t \searrow 0$, there exists $t_0 \in \mathbb{N}$ such that $x_t \leq \ln 2$ for all $t \geq t_0$. Using the inequalities

$$\exp(x) \geq 1 + x \quad \forall x \in \mathbb{R}, \quad \exp(-x) < 1 - x/2 \quad \forall x \in (0, \ln 2],$$

it follows that for all $t \geq t_0$,

$$\begin{aligned} \frac{p_{t+1}x_{t+1}}{p_{t_0}x_{t_0}} &\geq \prod_{s=t_0}^{t} \exp\left(-\frac{2\gamma x_s}{\alpha + \gamma y^H} - \frac{\gamma x_{s+1}}{\alpha + \gamma y^L} \right) \qquad (3.7) \\ &\geq \prod_{s=t_0}^{t} \exp\left(-\frac{3\gamma x_s}{\alpha + \gamma y^L} \right) \geq \exp\left(-\frac{3\gamma}{\alpha + \gamma y^L} \sum_{s=t_0}^{\infty} x_s \right). \end{aligned}$$

For $\gamma = 0$ (that is, for exponential utility),

$$\frac{p_{t+1}x_{t+1}}{p_0 x_0} = \prod_{s=0}^{t} \exp(-\alpha(x_s + x_{s+1})) \geq \prod_{s=0}^{t} \exp(-2\alpha x_s) \geq \exp(-2\alpha \sum_{s=0}^{\infty} x_s). \qquad (3.8)$$

Since $x_t \searrow 0$, (A.3) implies that there exists $0 < l < 1$ such that $\frac{x_{t+1}}{x_t} < l$ for all t large enough, which implies the convergence of the series $\sum x_t$. Therefore $(p_t x_t)$ is bounded away from zero and $\lim p_t x_t > 0$, by (3.7) and (3.8). □

Proposition 3.2 shows that for a large class of utility functions the discounted total debt limits do not vanish in the nonstationary equilibria of Proposition 3.1, and therefore bubbles can be sustained in equilibrium.[8]

The existing literature suggests that the endogeneous debt limits arising under the penalty (IT) (studied in this section) are more relaxed for high-income agents. Ábrahám and Cárceles-Poveda (2010) show this to be case for Markov equilibria, when each agent's income is iid over time. Alvarez and Jermann (2001) confirm this finding for the stationary equilibria in the setup of this paper. It is true that even in the nonstationary equilibria of Proposition 3.1, debt limits for high-types cannot be tighter then the debt limits of low-types at *all* periods. Otherwise, if $\phi_t^H \geq \phi_t^L$ for *all* t,

$$\frac{p_{t+1}}{p_t}\phi_{t+1}^H \geq \phi_t^H \geq \phi_t^L = \frac{p_{t+1}}{p_t}\phi_{t+1}^H,$$

and it follows that $(p_t\phi_t^H), (p_t\phi_t^L)$ are constant, and $\phi_t^L = \phi_t^H$ for all t. Nevertheless, Proposition 3.2 (with HARA utilities) implies that $\underline{L} < \bar{L}$ since $\lim p_t x_t > 0$. Therefore there is an indeterminacy of agents' debt limits, obtained by varying the initial wealth of the agents.[9]. Among these debt limits, there are some for which $\phi_t^H > \phi_t^L$ at *some* dates t. This remains true even when the initial wealth of the agents is fixed to zero (and there is a one period transition to the constrained equilibrium paths described in Proposition 3.1). The numerical example of Section 5 illustrates these issues.

For penalty (IB), Hellwig and Lorenzoni (2009) showed that in all non-autarchic constrained equilibria (stationary and nonstationary), the maximal initial size of a bubble equals the initial amount of trading (transfers) between agents. Thus one could conjecture that for a given penalty for default, if multiple equilibria are possible, the size of the bubbles they can sustain can be ranked by the amount of risk-sharing (trading) they allow. This is false for penalty (IT). Indeed, given equilibrium transfers $(x_t)_{t\geq 0}$ with $x_t \searrow 0$ with associated bond prices $(q_t)_{t\geq 0}$, the initial size of the bubble is given by $\lim p_t x_t = \lim q_0 q_1 \ldots q_{t-1} x_t$. Therefore the forward "shifted" equilibrium sequence of transfers $(x'_t)_{t\geq 0}$ given by $x'_t = x_{t+1}$ (that is, starting from $x'_0 := x_1$) leads to a bubble of initial size $\lim q_1 q_2 \ldots q_{t-1} x_t = \frac{1}{q_0} \lim p_t x_t$. If the initial level of transfer is x_0 is slightly larger than x^* (given by (2.19)), then $q_0 < 1$ (since at

[8]The HARA utility assumption in Proposition 3.2 simplifies the proof, and it can likely be relaxed.

[9]As seen in Proposition 2.1, martingales added to discounted debt limits leave agents' budget constraints unchanged if the initial wealth of the agent is increased by the initial value of the martingale.

constant transfers x^*, bond prices are 1), and transfers (x'_t) lead to a bubble greater than the one possible under transfers (x_t) (even though $x'_0 = x_1 < x_0$). On the other hand, if initial transfers x_0 are lower than x^*, then $q_0 > 1$ and the "shifted" sequence of transfers (x'_t) leads to a smaller bubble than (x_t).

Allowing for a one-period transition from a fixed level of initial wealth preserves this conclusion. Thus the size of bubbles in the first period (when the economy reaches the constrained equilibrium path) is not necessarily comonotonic with the amount of transfers in the same period. The numerical example of Section 5 makes this discussion concrete.

3.2 The stochastic case

The stochastic case ($\pi < 1$) can be reduced to the deterministic case analyzed before. Agents' continuation utilities after default are determined from

$$\begin{aligned} V^{H,d} &= u(y^H) + \beta\pi V^{L,d} + \beta(1-\pi)V^{H,d}, \\ V^{L,d} &= u(y^L) + \beta\pi V^{H,d} + \beta(1-\pi)V^{L,d}, \end{aligned}$$

which leads to

$$V^{H,d} = \frac{u(y^H) + \tilde{\beta}u(y^L)}{(1-\beta+\beta\pi)(1-\tilde{\beta}^2)}, \tag{3.9}$$

where $\tilde{\beta}$ was defined in (2.9). Similarly, agents' continuation utilities in a constrained equilibrium $((x_t),(a_t),(\phi^L_t))$ are

$$\begin{aligned} V^H(-a_t) &= u(y^H - x_t) + \beta\pi V^L(a_{t+1}) + \beta(1-\pi)V^H(-a_t), \\ V^L(a_t) &= u(y^L + x_t) + \beta\pi V^H(-a_{t+1}) + \beta(1-\pi)V^L(a_t), \end{aligned}$$

and therefore

$$V^H(-a_t) = \frac{u(y^H - x_t) + \tilde{\beta}u(y^L + x_{t+1})}{(1-\beta+\beta\pi)(1-\tilde{\beta}^2)}. \tag{3.10}$$

As high-types are indifferent between defaulting or not, $V^H(-a_t) = V^{H,d}$, hence

$$u(y^H - x_t) + \tilde{\beta}u(y^L + x_{t+1}) = u(y^H) + \tilde{\beta}u(y^L). \tag{3.11}$$

As a consequence, the equilibrium transfers are determined exactly as in the deterministic case, with $\tilde{\beta}$ replacing β in (3.1). Notice that the evolution of asset holdings (2.14) can be written as $x_t - \tilde{q}_t \tilde{a}_{t+1} = \tilde{a}_t$, where

$$\tilde{q}_t = \frac{\pi q_t^c}{1-\beta+\beta\pi} = \tilde{\beta}\frac{u'(y^L + x_{t+1})}{u'(y^H - x_t)}, \quad \tilde{a}_t = (1-\beta+\beta\pi)a_t. \tag{3.12}$$

It follows that $\big((x_t), (a_t), (\phi_t^L)\big)$ is a constrained equilibrium for the stochastic economy with agents' discount rate β if and only if $\Big((x_t), (\tilde{a}_t), (\tilde{\phi}_t^L)\Big)$ is an equilibrium for the deterministic economy with agents' discount rate $\tilde{\beta}$ given by (2.9), asset holdings $(\tilde{a}_t)$ satisfying (3.12), and debt limits

$$\tilde{\phi}_t^L = (1-\beta+\beta\pi)\phi_t^L, \quad \forall t \geq 0. \tag{3.13}$$

Let $\Big((x_t), (\tilde{a}_t), (\tilde{\phi}_t^L)\Big)$ be a nonstationary equilibrium for the deterministic economy with agents' discount rate $\tilde{\beta}$ (given by (2.9)). By Proposition 3.1 and 3.2, there are bubbles $(\tilde{b}_t)$ (satisfying $\tilde{b}_t = \tilde{q}_t \tilde{b}_{t+1}$) and such that $0 \leq \tilde{b}_t \leq -(\tilde{\phi}_t^H + \tilde{\phi}_t^L)$. The maximal such bubble is given by $\tilde{b}_0 = \lim_{t\to\infty} \tilde{q}_0 \tilde{q}_1 \dots \tilde{q}_{t-1} x_t$. By (3.12),

$$\tilde{b}_t = \pi q_t^c \tilde{b}_{t+1} + (1-\pi)\beta \tilde{b}_t. \tag{3.14}$$

Therefore $(\tilde{b}_t)$ is a discounted martingale process for the equivalent stochastic economy with agents' discount factors β, in the sense that $\tilde{b}_t$ is the value of the process in histories with t reversals. Scaling by constants preserves the martingale property, and therefore (b_t) given by $b_t := (1-\beta+\beta\pi)^{-1}\tilde{b}_t$ can be sustained as a bubble for the stochastic economy, since it satisfies $b_t \leq -(\phi_t^H + \phi_t^L)$. Hence any (deterministic) nonstationary equilibrium in Proposition 3.1 leads to an equivalent nonstationary stochastic equilibrium that can sustain bubbles.

4 Temporary interdiction to trade

I consider first the deterministic case when the penalty for default is the interdiction to trade for a finite number of periods (ITF). If the penalty for default is sufficiently mild (M small), there exists a stationary equilibrium with low interest rates that can sustain a bubble. Stochastic economies with one-period exclusion from the markets after default (that is, with penalty (ITF) with $M = 1$) are a particular case of (ITR), with $\mu = 1$ (agents reenter the markets with probability 1 in the period following default). For the stochastic case, therefore, I tackle directly the penalty (ITR), and show that for sufficiently large μ and π, there exists a stationary (stochastic) equilibrium with low interest rates that can sustain bubbles.

4.1 Interdiction to trade for a finite number of periods

This section analyzes the penalty (ITF) for the deterministic case ($\pi = 1$). For concreteness, M is assumed to be odd, thus $M = 2m + 1$, for some $m \geq 0$. The case when M is even can be analyzed in an identical way.

I give conditions on the parameters under which there exists a stationary equilibrium $((\hat{x}), (\hat{a}), (\hat{\phi}^L))$ with low interest rates that can sustain bubbles. In such an equilibrium, bond prices are $\hat{q} = q(\hat{x})$, where $q(\cdot)$ is defined as

$$q(x) := \beta u'(y^L + x)/u'(y^H - x). \tag{4.1}$$

Agents' budget constraints imply $\hat{q}\hat{a} + \hat{a} = \hat{x}$. The continuation utility of a high-type along the equilibrium path is $V^H(-\hat{a}) = (u(y^H - \hat{x}) + \beta u(y^L + \hat{x}))/(1 - \beta^2)$. Along the stationary equilibrium path, a low-type is constrained even though he starts the period with positive wealth $\hat{a}$, and therefore he will be also constrained starting with zero wealth after he defaults. It follows that

$$V^{H,d} = u(y^H) + \beta u(y^L) + \beta^2 u(y^H) + \ldots + \beta^{2m} u(y^H) + \beta^{2m+1} u(y^L + \hat{q}\hat{a}) + \beta^{2m+2} V^H(-\hat{a}).$$

Since high-types are indifferent between defaulting or not, $V^H(-\hat{a}) = V^{H,d}$, therefore the transfer $\hat{x}$ satisfies $g(\hat{x}; m) = 0$, where

$$\begin{aligned} g(x; m) := -u(y^H - x) - \beta u(y^L + x) + u(y^H) + \beta u(y^L) + \\ + \frac{\beta^{2m} - \beta^{2m+2}}{1 - \beta^{2m+2}} \left(u(y^H) + \beta u \left(y^L + \frac{q(x)x}{1 + q(x)} \right) - u(y^H) - \beta u(y^L) \right). \end{aligned} \tag{4.2}$$

Notice that $g(0; m) = 0$, which, incidentally, shows that autarchy is an equilibrium. Moreover, for any $x > 0$, $u(y^H) + \beta u(y^L) < u(y^H) + \beta u \left(y^L + \frac{q(x)x}{1+q(x)} \right)$, hence $g(x; m) < g(x; 0)$ and an increase in m (the length of exclusion) shifts the function $g(x; m)$

downward (at all points $x > 0$). Notice that

$$g'(0;m) = \frac{u'(y^H)}{1+q(0)} \left(1 - \frac{1-\beta^{2m}}{1-\beta^{2m+2}} q^2(0)\right),$$

where $q(0) = \beta u'(y^L)/u'(y^H) > 1$, by (2.9). I assume that $g'(0;m) > 0$, which is equivalent to requiring that the duration of exclusion is sufficiently short:

$$\beta^{2m} > 1 - \frac{1-\beta^2}{q^2(0) - \beta^2}. \tag{4.3}$$

For a one-period interdiction to trade upon default ($M = 2m+1 = 1$), (4.3) holds.

I also impose the condition that $g(x^*;0) < 0$ (x^* is given by (2.19)), which amounts to

$$u(y^H) + \beta u(y^L + x^*/2) < u(y^H - x^*) + \beta u(y^L + x^*). \tag{4.4}$$

For (2.19) to hold, u needs to be sufficiently concave and y^L sufficiently small compared to y^H.[10] Conditions (4.3) and (4.4) guarantee the existence of an $\hat{x} \in (0, x^*)$ such that $g(\hat{x};m) = 0$. This $\hat{x}$ represents the level of transfers in a stationary equilibrium with with low interest rates, as $\hat{q} > 1$, since $\hat{x} < x^*$. This discussion is summarized as follows.

Proposition 4.1. *Assume that* (4.3) *and* (4.4) *hold. There exists a stationary equilibrium with transfers* $\hat{x} \in (0, x^*)$ *such that* $g(\hat{x};m) = 0$, *asset holdings* $-\hat{a} := -\hat{x}/(1+\hat{q})$ *and some debt limits* $\hat{\phi}^L < 0$ *for the low-type.*

Incidentally, notice that the equilibrium in Proposition 4.1 is "robust" to the introduction of small bankruptcy costs, in the language of Azariadis and Kaas (2013).[11]

[10] If, for example, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and $y^L = 0$, it can be checked that for $\gamma = 2$, $g(x^*;0) = -\beta/y^H < 0$, while for $\gamma = 3$, $g(x^*;0) = -(3 + \beta^{1/3} + 5\beta^{2/3})/(\sqrt{2}y^H)^2 < 0$.

[11] Assume that a defaulting agent looses also a small $\varepsilon > 0$ out of his endowment in the default period in addition to the interdiction to trade for M periods. Let $g_\varepsilon(x;m) := g(x,m) - \frac{1-\beta^2}{1-\beta^{2m+2}}(u(y^H) - u(y^H - \varepsilon))$ capturing the downward shift in g due to the additional bankruptcy cost for default. Notice that $\frac{\partial g_\varepsilon(x;m)}{\partial y^H} \neq 0$. Therefore by the transversality theorem and the implicit function theorem, for a generic set of parameters β, y^H, y^L, for all small ε there exists a zero of g_ε, denoted by $\hat{x}(\varepsilon)$, which is close to $\hat{x}$ and represents an equilibrium level of transfers for the economy with bankruptcy costs. As $\varepsilon \to 0$, $\hat{x}(\varepsilon) \to \hat{x}$.

The debt limit $\hat{\phi}^L$ of low-types is calculated in Proposition A.1 and shown to belong to $(-\hat{q}\hat{a}, 0)$. Therefore $\hat{\phi}^L > \hat{q}\hat{\phi}^H$, since $\hat{\phi}^H = -\hat{a}$. If (b_t) is a martingale component of the debt limits of the initial low-type, it must be the case that $b_t = \hat{q}^t b_0$ and $b_0 \leq -\hat{\phi}^L$, $b_1 \leq -\hat{\phi}^H$, $b_2 \leq -\hat{\phi}^L$ etc. As $\hat{q} > 1$, these inequalities are equivalent to $b_0 \leq \min\{-\hat{\phi}^L, -\hat{\phi}^H/\hat{q}\}$. A similar reasoning for the initial high-type shows the maximal initial martingale component is $\min\{-\hat{\phi}^H, -\hat{\phi}^L/\hat{q}\} = -\hat{\phi}^L/\hat{q}$. Proposition 2.1 guarantees that bubbles of initial size

$$\min\{-\hat{\phi}^L, -\hat{\phi}^H/\hat{q}\} + \min\{-\hat{\phi}^H, -\hat{\phi}^L/\hat{q}\} \leq -\hat{\phi}^L - \hat{\phi}^H < -\hat{q}\hat{\phi}^H - \hat{\phi}^H = \hat{x}$$

can be sustained in equilibrium. As $\hat{x} < x^* < \min\{\bar{x}, (y^H - y^L)/2\}$, the equilibrium in Proposition 4.1 associated to punishment (ITF) sustains both less risk sharing and smaller initial bubbles than the stationary equilibria under penalties (IB) or (IT).

4.2 Interdiction to trade for a random number of periods

Consider a stationary equilibrium $(x, a(x), \phi^L(x))$ for the general stochastic economy $(\pi \leq 1)$ with penalty (ITR), and let $q^c(x) = \beta u'(y^L + x)/u'(y^H - x)$ and $q^{nc}(x) = \beta$ be the prices of the Arrow securities. The continuation utilities after default satisfy

$$\begin{aligned} V^{H,d} &= u(y^H) + \beta\mu\left(\pi V^L(0) + (1-\pi)V^H(0)\right) + \beta(1-\mu)\left(\pi V^{L,d} + (1-\pi)V^{H,d}\right), \\ V^{L,d} &= u(y^L) + \beta\mu\left(\pi V^H(0) + (1-\pi)V^L(0)\right) + \beta(1-\mu)\left(\pi V^{H,d} + (1-\pi)V^{L,d}\right). \end{aligned}$$

To calculate $V^L(0)$, notice that a low-type starting with 0 borrows the maximal amount allowed if the state switches (as he was also borrowing the maximal amount along the equilibrium path, starting with wealth $a(x) > 0$). If the state does not switches, he borrows zero (maintains his wealth and consumption). Clearly these choices satisfy the necessary and sufficient Euler conditions for the low-type, hence

$$V^L(0) = u(y^L + \pi q^c(x)a(x)) + \beta\left(\pi V^H(-a(x)) + (1-\pi)V^L(0)\right). \tag{4.5}$$

To solve for $V^H(0)$, I analyze next the optimal choices of a high-type agent starting with wealth 0 ($\geq -a(x)$) at some period t. He consumes $y^H - x_0(x)$ at t, saves $a_1(x)$ ($\geq a(x)$) contingent on a reversal occurring (that is, if $s_{t+1} \neq s_t$), and saves 0 if there is no state change ($s_{t+1} = s_t$):

$$V^H(0) = u(y^H - x_0(x)) + \beta\pi V^L(a_1(x)) + \beta(1-\pi)V^H(0).$$

I look at equilibria where the low-type starting with wealth $a_1(x)$ consumes some $y^L + x_1(x)$ and has binding debt limits in the high endowment state next period:

$$V^L(a_1(x)) = u(y^L + x_1(x)) + \beta\pi V^H(-a(x)) + \beta(1-\pi)V^L(a_1(x)).$$

Transfers $x_0(x), x_1(x)$ can be expressed in terms of $a_1(x)$ using agent's budgets at t and $t+1$:

$$x_0(x) = \pi q^c(x) a_1(x),\ x_1(x) = a_1(x) + \pi q^c(x) a(x) - (1-\pi)\beta a_1(x). \tag{4.6}$$

Wealth $a_1(x)$, in turn (and therefore $x_0(x), x_1(x)$), is determined as the unique solution of agent's Euler equation at t conditional on a state change from t to $t+1$,

$$\beta \frac{u'(y^L + x_1(x))}{u'(y^H - x_0(x))} = q^c(x). \tag{4.7}$$

The Euler equations at t and $t+1$ conditional on the state not changing next period are satisfied (by construction). The described choices are therefore optimal for the high-type with zero wealth at t if the Euler equation at $t+1$ conditional on state change from $t+1$ to $t+2$ is satisfied,

$$\beta \frac{u'(y^H - x)}{u'(y^L + x_1(x))} \leq q^c(x). \tag{4.8}$$

Condition (4.8) will be shown to hold below. It follows that $V^L(0), V^H(0)$ can be expressed in terms of $V^H(-a(x))$:

$$V^L(0) = \frac{u(y^L + x_0(x)) + \beta\pi V^H(-a(x))}{1 - \beta + \beta\pi},$$

$$V^H(0) = \frac{(1-\beta+\beta\pi)u(y^H - x_0(x)) + \beta\pi u(y^L + x_1(x)) + (\beta\pi)^2 V^H(-a(x))}{(1-\beta+\beta\pi)^2}.$$

The continuation utility after default for high-types is

$$V^{H,d} = \frac{1}{2}\underbrace{\frac{u(y^H)+u(y^L)+\beta\mu(V^L(0)+V^H(0))}{1-\beta(1-\mu)}}_{V^{H,d}+V^{L,d}} + \tag{4.9}$$
$$+\frac{1}{2}\underbrace{\frac{u(y^H)-u(y^L)+\beta\mu(1-2\pi)(V^H(0)-V^L(0))}{1-\beta(1-\mu)(1-2\pi)}}_{V^{H,d}-V^{L,d}}.$$

Finally, the continuation utility of high-types is given by (see (3.10))

$$V^H(-a(x)) = \frac{u(y^H-x)+\tilde{\beta}u(y^L+x)}{(1-\beta+\beta\pi)(1-\tilde{\beta}^2)}. \tag{4.10}$$

Since high-types are indifferent between defaulting or not, $V^{H,d}$ in (4.9) must equal $V^H(-a(x))$ in (4.10). Therefore x is indeed an equilibrium level of transfers if

$$0 = h(x,\pi,\mu) := V^{H,d} - V^H(-a(x)). \tag{4.11}$$

The case $\pi = 1$, $\mu = 1$ amounts to a deterministic economy with one period exclusion from the markets after default ((ITF) with $M = 2m+1 = 1$). Therefore $h(x,1,1) = g(x;0)$ with g given by (4.2).

Under the assumption (4.4) (as (4.3) is automatically satisfied for $m = 0$), the deterministic economy with penalty (ITF) with $M = 2m+1 = 1$ admits a stationary equilibrium with transfers $\hat{x}$ satisfying $h(\hat{x},1,1) = g(\hat{x};0) = 0$ (Proposition 4.1). Notice that for any $x > 0$, $\frac{\partial g(x;0)}{\partial \beta} \neq 0$, $\frac{\partial g(x;0)}{\partial y^L} \neq 0$ and $\frac{\partial g(x;0)}{\partial y^H} \neq 0$. By the transversality theorem, $\frac{\partial g(\hat{x};1)}{\partial x} \neq 0$, for a *generic* set of parameters β,y^L or y^H. By the implicit function theorem, for μ,π sufficiently close to 1, there exists $x(\pi,\mu)$ close to $\hat{x}$ such that $h(x(\pi,\mu),\pi,\mu) = 0$. It follows that $x_0(x(\pi,\mu))$,$x_1(x(\pi,\mu))$ are also close to $\hat{x}$ and therefore the condition (4.8) required for the described allocations to be indeed an equilibrium is satisfied, as $\hat{q} = q^c(\hat{x}) > 1$ and

$$\beta\frac{u'(y^H - x(\pi,\mu))}{u'(y^L + x_1(x(\pi,\mu))} \approx \beta\frac{u'(y^H-\hat{x})}{u'(y^L+\hat{x})} = \frac{\beta^2}{\hat{q}} < \hat{q} \approx q^c(x(\pi,\mu)).$$

For π,μ sufficiently close to 1, $\pi q^c(x(\pi,\mu)) + (1-\pi)\beta > 1$ and therefore risk-free rates are low. This implies discounted debt limits have martingale components, since, in particular, a deterministic bubble of initial size $\min\{|\phi^L|, |\phi^H|\}$ can be sustained in equilibrium.

5 Numerical example

As a numerical illustration, let $y^L = 1$, $y^H = 2$, $\beta = 0.99$, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $\gamma = 3$, and $\pi = 1$ (deterministic economy). For penalty (IB), the stationary equilibrium transfer level is $x^* \approx 0.4975$. As described in Section 2.4, for any initial transfer $0 < x_0 < x^*$, there are nonstationary equilibria (x_t) converging monotonically to autarchy, $x_t \searrow 0$. Agents' discounted debt limits are *constant* (martingales) and these equilibria can sustain bubbles of initial size equal to the initial transfer x_0 (Hellwig and Lorenzoni 2009).

For penalty (IT), notice that (3.5) does not hold. Therefore the stationary equilibrium displays perfect risk sharing and transfers from high to low-types are $\frac{1}{2}$. In the stationary equilibrium, interest rates are high (bond prices are $q^c = \beta = 0.99$) and bubbles cannot exist. Debt limits are

$$\phi^L = -\frac{0.5\beta}{1+\beta} > \phi^H = -\frac{0.5}{1+\beta}.$$

In the nonstationary equilibrium with initial transfers $x_0 = 0.5$, $\lim_{t\to\infty} p_t x_t \approx 0.243967$. Initial holdings a_0 are arbitrary in $[\underline{L}, \bar{L}] = [0.2258, 0.4697]$. Debt limits vary over time, depend on the initial holdings and do not have to be more relaxed for high-types at all periods. For example, if a_0 is close to $\underline{L}$, say $a_0 = 0.24$, then $\phi_0^H = -a_0 = -0.24 > -0.5 + 0.24 = -x_0 - \phi_0^H = \phi_0^L$.

The (period zero) size of the bubble is not comonotonic with the initial amount of risk sharing (initial transfers) across the different nonstationary equilibria. Indeed, for $x_0 = x^* = 0.4975$, $a_0 \in [0.2236, 0.4675]$ and $\lim_{t\to\infty} p_t x_t \approx 0.243972$, hence a bigger bubble is possible than the one associated to the equilibrium with (higher) initial transfers 0.5.[12] On the other hand, for small initial transfers $x_0 < x^*$, the initial size of the bubble is small, as $\lim p_t x_t < x_0$. For example, for $x_0 = 0.2$, $a_0 \in [0.0373, 0.1952]$ and $\lim p_t x_t = 0.157901$. Notice that even if agents' initial wealth is fixed and equal to 0, the three nonstationary equilibria associated to $x_0 \in$

[12]Similarly, for $x_0 = \bar{x}' = 0.7332$ (see (3.3)), $\lim p_t x_t = 0.190467$, leading to a lower bubble than in the equilibria associated with initial transfers x^* or 0.5.

$\{0.5, 0.4975, 0.2\}$ can be reached after a one period transition, as (2.18) holds, and the (non-comonotonic) relation between the size of the bubbles and the amount of risk sharing (in the period when the transition ends) is preserved.

For the (ITF) with $M = 2m + 1 = 1$ (thus (4.3) is satisfied), or equivalently for (ITR) with $\mu = 1$, the chosen parameters satisfy (4.4). The equilibrium transfers, bond prices and asset holdings are $\hat{x} \approx 0.4271$, $\hat{q} \approx 1.3255$ and $\hat{a} \approx 0.1837$. The debt limits of the low-type are obtained using Proposition A.1, $\phi^L \approx -0.1972$. The debt limits of the high type are $\phi^H = -\hat{a} \approx -0.1837$. Notice that the debt limits of the high-types are in fact *tighter* than the debt-limits of low-types. As $\phi^L > \hat{q}\phi^H$ and $\phi^H > \hat{q}\phi^L$, agents' discounted debt limits are *decreasing*. Therefore the maximum initial size of a bubble that can be sustained in equilibrium is $-\phi^L - \phi^H = 0.3808$. In contrast to penalties (IB) and (IT), there are multiple stationary equilibria. It can verified that transfers $(y^H - y^L)/2$ lead to an (unconstrained) *stationary* equilibrium with perfect risk-sharing between agents. Indeed, $g(\frac{y^H - y^L}{2}; 0) < 0$ (with g defined in (4.2)), which is the necessary and sufficient condition for perfect risk-sharing (first best allocation) to be an equilibrium (Azariadis and Kaas 2008, Proposition 1). Interest rates are high in this equilibrium as bond prices are $q_t = \beta < 1$, and no bubbles are possibles.

A comparison of the stationary equilibria for *different* penalties for default reveals that the amount of risk sharing and the size of sustainable bubbles are not comonotonic. Indeed, the amount of risk sharing in these stationary equilibria is largest for (IT) and smallest for (ITF), as

$$\frac{y^H - y^L}{2} = 0.5 > x^* = 0.4975 > \hat{x} = 0.4271.$$

On the other hand, the size of bubbles in the described (non-autarchic) stationary equilibria under penalties (IT), (IB), (ITF) are 0, $x^* = 0.4975$ and 0.3808. This comparison is robust to allowing for one period transition from fixed, zero initial wealth for the agents. Indeed, it is immediate to check that (2.18) holds for the three stationary equilibria considered here.

6 Conclusion

This paper argues that low interest rates and bubbles are prevalent in economies with limited enforcement of debt contracts. With mild penalties for default such as a permanent or temporary interdiction to trade, interest rates adjust endogenously to a low level to prevent default, and bubbles can be sustained in equilibrium. This complements the earlier results of Hellwig and Lorenzoni (2009) obtained for an interdiction to borrow as penalty for default.

Bubbles here serve as a substitute for private liquidity. They are associated to (dynamic) inefficiencies only insofar as the enforcement limitations induce inefficient levels of interest rates and risk sharing in the absence of bubbles. Farhi and Tirole (2012) showed that financial frictions can sever the connection between (dynamic) inefficiency and low interest rates, allowing for (some) bubbly equilibria to be efficient. They make this point in an economy with limited pledgeability, where debt is fully collateralized, rather than being sustained by reputation.

For penalties (IB),(ITF),(ITR), the continuation utilities after default depend on endogenous equilibrium variables such as prices and debt limits, and therefore a definition of constrained inefficiency is not obvious. Adopting the definitions of efficiency given in Bloise and Reichlin (2011), Appendix B gives a detailed discussion of the efficiency of the various equilibria constructed here. All but the stationary equilibrium associated to penalty (IB) are inefficient.

A Omitted proofs

Proof of Proposition 3.1

Proof. Function $\bar{f}(x) := f(x, x)$ is (strictly) convex, hence $\bar{f}'$ is strictly increasing. Moreover, $\bar{f}(0) = 0$, $\bar{f}(y^H - y^L) > 0$, $\bar{f}'((y^H - y^L)/2) > 0$ and $\bar{f}'(0) < 0$ (by (2.9)). The function $\bar{f}$ decreases strictly up to x^* given by $\bar{f}'(x^*) = 0$ (and hence by (2.19)) and then increases strictly. It follows that there exists a (unique) $\bar{x} \in (x^*, y^H - y^L)$, such that $\bar{f}(\bar{x}) = f(\bar{x}, \bar{x}) = 0$, which is equivalent to (3.2).

For $x_t \in (0, \bar{x})$, $f(x_t, 0) > 0$, $f(x_t, x_t) = \bar{f}(x_t) < 0$ and $f(x_t, \cdot)$ is strictly decreasing. It follows that the equation $f(x_t, x_{t+1}) = 0$ has a unique solution $x_{t+1} := x^{next}(x_t)$, which moreover satisfies $0 < x_{t+1} < x_t$. Therefore the sequence

(x_t) satisfying $f(x_t, x_{t+1}) = 0$ for all t is strictly decreasing if $0 < x_0 < \bar{x}$. Moreover, the continuity of f implies that $f(\lim x_t, \lim x_t) = \bar{f}(\lim x_t) = 0$, and thus $\lim x_t = 0$. If $x_t \in \{0, \bar{x}\}$, then $f(x_t, x_{t+1}) = 0$ if and only if $x_{t+1} = x_t$. Hence for $x_0 \in \{0, \bar{x}\}$, $x_t = x_0$ for all t.

Let $\tilde{f}(x) := u(y^H) + \beta u(y^L) - (1+\beta)u(y^H - x)$, which is increasing on $[0, y^H - y^L]$. Notice that $\tilde{f}(x) > (<)\bar{f}(x)$ for $x < (>)\frac{y^H - y^L}{2}$. It follows that $\frac{y^H-y^L}{2} > \bar{x}' > \bar{x}$ or $\frac{y^H-y^L}{2} \le \bar{x}' \le \bar{x}$, with $\bar{x}'$ defined in (3.3). As a consequence, the first order conditions (2.13) of the low-type agents hold for any $x_0 \le \min\{\bar{x}, \bar{x}'\}$.

A low-type agent with starting wealth ϕ_t^L at t instead of a_t will again be borrowing constrained. The not-too-tight condition for ϕ_t^L and agent's budget constraint at t imply that

$$\phi_t^L = -\frac{p_{t+1}}{p_t} a_{t+1} = \frac{p_{t+1}}{p_t} \phi_{t+1}^H. \tag{A.1}$$

By (2.14) and (2.15),

$$\phi_t^H + \phi_t^L = -a_t - \frac{p_{t+1}}{p_t} a_{t+1} = -x_t. \tag{A.2}$$

Next we determine the restrictions needed on the initial wealth of the agents such that the debt bounds are nonpositive, or equivalently, the asset holdings (a_t) given by (2.21) are nonnegative. This is clearly the case when $x_t = 0$ for all t (the autarchic equilibrium), since asset holdings and debt bounds are zero. For non-autarchic equilibria, that is for nonzero sequences (x_t), by (3.1) and the strict concavity of u,

$$u'(y^H - x_t)x_t > u(y^H) - u(y^H - x_t) = \beta\left(u(y^L + x_{t+1}) - u(y^L)\right) > \beta u'(y^L + x_{t+1})x_{t+1}.$$

Therefore by (2.11),

$$\frac{p_t}{p_{t+1}} = \frac{u'(y^H - x_t)}{\beta u'(y^L + x_{t+1})} > \frac{x_{t+1}}{x_t}. \tag{A.3}$$

It follows that the sequence $(p_t x_t)$ is strictly decreasing. By (A.2),

$$p_t x_t = -p_t \phi_t^H - p_t \phi_t^L > p_{t+1} x_{t+1} = -p_{t+1}\phi_{t+1}^H - p_{t+1}\phi_{t+1}^L,$$

and using (A.1),

$$p_t \phi_t^H < p_{t+1} \phi_{t+1}^L. \tag{A.4}$$

By (A.1) and (A.4), for $i \in I$, the sequences $(p_t \phi_t^i)$ are nondecreasing. As a consequence, the necessary and sufficient condition for $\phi^i \leq 0$ is $\lim_{t\to\infty} p_t\phi_t^i \leq 0$. Since $(p_t x_t)$ is decreasing, the limits $\underline{L}$ and $\bar{L}$ exist and $\bar{L} = \underline{L} + \lim_{t\to\infty} p_t x_t$. By (2.21),

$$\lim_{t\to\infty} p_t\phi_t^1 = -\lim_{t\to\infty} p_{2t+1}a_{2t+1} = a_0 - \bar{L}, \quad \lim_{t\to\infty} p_t\phi_t^2 = -\lim_{t\to\infty} p_{2t}a_{2t} = -a_0 + \underline{L}.$$

Therefore $\phi^1, \phi^2 \leq 0$ if and only if $\underline{L} \leq a_0 \leq \bar{L}$.

Finally, agents' transversality conditions are satisfied. Indeed, for $i \in I$,

$$\lim_{t\to\infty} \beta^t u'(c_t^i)(a_t^i - \phi_t^i) \leq \lim_{t\to\infty} \beta^t u'(c_t^i)\left(\sum_i a_t^i - \sum_i \phi_t^i\right) = \lim_{t\to\infty} \beta^t u'(c_t^i)x_t = 0. \quad \text{(A.5)}$$

□

Proposition A.1. *Consider the equilibrium described in Proposition 4.1. Let c^H, c^L be (uniquely) determined by*

$$\beta\frac{u'(c^L)}{u'(c^H)} = \hat{q}, \quad c^H + \hat{q}c^L = y^H - \hat{x} + \hat{q}(y^L + \hat{x}) + \hat{a}. \quad \text{(A.6)}$$

If

$$\beta\frac{u'(y^H - \hat{x})}{u'(c^L)} \leq \hat{q}, \quad \text{(A.7)}$$

then ϕ^L satisfies

$$u(y^L + \phi^L + \hat{q}\hat{a}) = u(y^L) + \beta(1-\beta^{2m})\frac{u(y^H) + \beta u(y^L))}{1-\beta^2} + \quad \text{(A.8)}$$
$$+\beta^{2m+1}u(c^H) + \beta^{2m+2}u(c^L) - \beta(1-\beta^{2m+2})\frac{u(y^H - \hat{x}) + \beta u(y^L + \hat{x})}{1-\beta^2}.$$

Proof. A low-type agent starting with wealth $\phi^L < \hat{a}$ at some date t instead of $\hat{a}$ (but facing the same future debt limits alternating between ϕ^H and ϕ^L and bond prices $\hat{q}$) will find optimal to borrow the maximum amount allowed $-\hat{a} = \phi^H$. Therefore

$$V^{L,d} = u(y^L + \phi^L + \hat{q}\hat{a}) + \beta V^H(-\hat{a}) > u(y^L + \phi^L + \hat{q}a) + \beta V^H(-a), \forall a < \hat{a}, \quad \text{(A.9)}$$

where

$$V^{L,d} = u(y^L) + \beta u(y^H) + \ldots + \beta^{2m}u(y^L) + \beta^{2m+1}V^H(0) \quad \text{(A.10)}$$

is the continuation utility after default of a low-type. If $\phi^L \geq 0$, setting $a = 0$ in (A.9) leads to $V^{L,d} > u(y^L) + \beta V^H(0) \geq V^{L,d}$, which is a contradiction. Hence $\phi^L < 0$. Rewrite (A.9) as

$$0 = \zeta(\phi^L) := u(y^L + \phi^L + \hat{q}\hat{a}) + \beta V^H(-\hat{a}) - V^{L,d}. \tag{A.11}$$

Notice that $V^H(-\hat{a}) = (u(y^H - \hat{x}) + \beta u(y^L + \hat{x}))/(1 - \beta^2)$ while $V^H(0)$ (and hence $V^{L,d}$) is nonincreasing in ϕ^L, as tightening future debt limits cannot increase agent's optimal continuation utility. Therefore ζ is strictly increasing. Moreover, since

$$V^H(-\hat{a}) = V^{H,d} = u(y^H) + \beta u(y^L) + \ldots + \beta^{2m} u(y^H) + \beta^{2m+1} V^L(0),$$

we get

$$\zeta(\phi^L) = u(y^L + \phi^L + \hat{q}\hat{a}) + \beta^{2m+1} u(y^H) + \beta^{2m+2} V^L(0) - u(y^L) - \beta^{2m+1} V^H(0).$$

Therefore $\zeta(-\hat{q}\hat{a}) = \beta^{2m+1}(u(y^H) + \beta V^L(0) - V^H(0)) < 0$, and we conclude that $\phi^L \in (-\hat{q}\hat{a}, 0)$.

Under (A.7), a high-type starting with zero wealth (rather than $-\hat{a}$) will borrow the maximum amount allowed. In the initial period t (with high endowment) he consumes some $c^H > y^H - \hat{x}$, and at $t + 1$ he consumes some $c^L > y^L + \hat{x}$, and afterwards (date $t + 2$ onwards) he reverts to consumption levels driven by transfers $\hat{x}$. Consumption levels c^H, c^L are determined from the first order conditions at t and from the intertemporal budget from t to $t + 2$, thus they are given by (A.6). The proposed consumption path is indeed optimal for the agent since the first order condition for date $t + 2$ (when low-type) holds, by (A.7). It follows that,

$$V^H(0) = u(c^H) + \beta u(c^L) + \beta^2 (u(y^H - \hat{x}) + \beta u(y^L + \hat{x}))/(1 - \beta^2). \tag{A.12}$$

Using (A.10) and (A.12), equation (A.11) which determines ϕ^L is equivalent to (A.8).

□

B Efficiency

B.1 Permanent interdiction to trade

I discuss first the efficiency of the equilibria in Proposition 3.1 for penalty (IT). An allocation $c = (c^1, c^2) \in X_+^I$ is *feasible* if $c_t^1 + c_t^2 = y_t^1 + y_t^2 (= y^H + y^L)$ for all t, and *individually rational* if $U_t^i(c^i) \geq U_t^i(y^i)$, for all $t \in \mathbb{N}$ and $i \in I$. An allocation $\bar{c}$ *Pareto dominates* allocation c if $U^i(\bar{c}^1) \geq U^i(c^1)$ for $i \in I$, with at least one strict inequality. A feasible and individually rational allocation c is *constrained inefficient* if it is Pareto dominated by another feasible and individually rational allocation $\bar{c}$ (Alvarez and Jermann 2000). An allocation c is *ex-post inefficient* if it is Pareto dominated by an allocation $\bar{c}$ satisfying $U_t^i(\bar{c}^i) \geq U_t^i(c^i)$ and $\sum_i \bar{c}_t^i \leq \sum_i c_t^i$, for all $t \in \mathbb{N}$ and $i \in \{e, o\}$. Conversely, an allocation is *constrained efficient* (respectively *ex-post efficient*), if it is *not* constrained inefficient (respectively ex-post inefficient). Notice that a feasible and individually rational allocation which is ex-post inefficient is always constrained inefficient.

Each nonstationary equilibrium of Proposition 3.1 associated to a sequence of transfers $x_t \to 0$ has the property that (by (2.9))

$$\frac{p_{t+1}}{p_t} \to \frac{\beta u'(y^L)}{u'(y^H)} > 1,$$

and therefore it satisfies the "modified Cass criterion", which is a sufficient condition for ex-post inefficiency (Bloise and Reichlin 2011, Lemma 2). Therefore all the non-stationary equilibria constructed in Proposition 3.1 are also constrained inefficient. By contrast, the stationary equilibrium is always constrained efficient. Indeed, if (3.5) is violated, the stationary equilibrium associated to transfers $(y^H - y^L)/2$ is actually Pareto optimal. If, instead, (3.5) holds, then in the stationary equilibrium associated to transfers $\bar{x}$ (see discussion after equation (3.5)),

$$\frac{p_{t+1}}{p_t} = \frac{\beta u'(y^L + \bar{x})}{u'(y^H - \bar{x})} < 1.$$

Therefore the stationary equilibrium violates the "weak modified Cass criterion", which is a necessary condition for constrained inefficiency (Bloise and Reichlin 2011, Lemma 3).

Based on this discussion, it is tempting to equate equilibrium low interest rates with inefficiency of the equilibrium. However the equivalence between efficiency of an equilibrium and the presence of high interest rates is not true in general and is a consequence of the stationarity of agents' endowments, as pointed out by Bloise and Reichlin (2011, Appendix B). They construct an efficient stationary equilibrium with low interest rates, in a framework similar to ours, but with nonstationary endowments.

B.2 Interdiction to borrow. Temporary interdiction to trade

I analyze here the efficiency of the equilibria associated to penalty (IB) (constructed in Hellwig and Lorenzoni (2009) and described in detail in Section 2.4) and of the equilibrium in Proposition 4.1 for penalty (ITF). The penalties for default now depend on endogenous equilibrium variables such as prices and debt limits, and therefore a definition of constrained inefficiency is not obvious. Following Bloise and Reichlin (2011), an allocation $c = (c^1, c^2) \in X_+^I$ is said to be *individually rational given reservation utilities* $\nu = (\nu^1, \nu^2) \in X^2$ if $U_t^i(c^i) \geq \nu_t^i$, for all $t \in \mathbb{N}$ and $i \in I$. A feasible allocation c is *constrained inefficient given some reservation utilities* $\nu \in X$ if it is Pareto dominated by an allocation $\bar{c}$ which is feasible and individually rational given the reservation utilities ν. The nonstationary equilibria for penalty (IB) described in Section 2.4 and the stationary equilibrium of Proposition 4.1 for penalty (ITF) are ex-post inefficient, by the modified Cass criterion, as bond prices $p_{t+1}/p_t > 1$ for large enough t. The stationary equilibrium for penalty (IB), associated to constant transfers x^* and zero risk-free interest rates (constant pricing kernel), is *not* constrained inefficient given reservation utilities $(V^{i,d})_{i\in I}$. This follows using an identical argument to the one used by Bloise and Reichlin (2011, Appendix B, Claims 5 and 7).

References

Abel, A., G. Mankiw, L. Summers, and R. Zeckhauser (1989): "Assessing Dynamic Efficiency: Theory and Evidence," *Review of Economic Studies*, 56(1), 1–19.

Ábrahám, A., and E. Cárceles-Poveda (2010): “Endogenous trading constraints with incomplete asset markets,” *Journal of Economic Theory*, 145(3), 974–1004.

Alvarez, F., and U. J. Jermann (2000): “Efficiency, Equilibrium, and Asset Pricing with Risk of Default,” *Econometrica*, 68(4), 775–797.

——— (2001): “Quantitative Asset Pricing Implications of Endogenous Solvency Constraints,” *Review of Financial Studies*, 14(4), 1117–1151.

Antinolfi, G., C. Azariadis, and J. B. Bullard (2007): “Monetary policy as equilibrium selection,” *Federal Reserve Bank of St. Louis Review*, July, 331–342.

Azariadis, C., and L. Kaas (2008): “Endogenous credit limits with small default costs,” mimeo, Washington University.

——— (2013): “Endogenous credit limits with small default costs,” *Journal of Economic Theory*, 148(2), 806–824.

Bidian, F. (2011): “Essays on Asset Price Bubbles,” Ph.D. thesis, University of Minnesota.

Bidian, F., and C. Bejan (2012): “Martingale Properties of Self-Enforcing Debt,” Working Paper 2012-04, CEAR, Georgia State University.

Bloise, G., and P. Reichlin (2011): “Asset prices, debt constraints and inefficiency,” *Journal of Economic Theory*, 146(4), 1520–1546.

Bloise, G., P. Reichlin, and M. Tirelli (2013): “Fragility of Competitive Equilibrium with Risk of Default,” *Review of Economic Dynamics*, 16(2), 271–295.

Bulow, J., and K. Rogoff (1989): “Sovereign Debt: Is to Forgive to Forget?,” *American Economic Review*, 79(1), 43–50.

Farhi, E., and J. Tirole (2012): “Bubbly Liquidity,” *Review of Economic Studies*, 79(2), 678–706.

Geerolf, F. (2013): “Reassessing Dynamic Efficiency,” mimeo, Toulouse School of Economics.

HELLWIG, C., AND G. LORENZONI (2009): "Bubbles and Self-Enforcing Debt," *Econometrica*, 77(4), 1137–1164.

HUANG, K. X. D., AND J. WERNER (2000): "Asset Price Bubbles in Arrow-Debreu and Sequential Equilibrium," *Economic Theory*, 15(2), 253–278.

KEHOE, T. J., AND D. K. LEVINE (1993): "Debt-Constrained Asset Markets," *Review of Economic Studies*, 60(4), 865–888.

——— (2001): "Liquidity Constrained Markets versus Debt Constrained Markets," *Econometrica*, 69(3), 575–598.

KOCHERLAKOTA, N. R. (1992): "Bubbles and Constraints on Debt Accumulation," *Journal of Economic Theory*, 57(1), 245–256.

——— (2008): "Injecting Rational Bubbles," *Journal of Economic Theory*, 142(1), 218–232.

LEROY, S., AND J. WERNER (2001): *Principles of financial economics*. Cambridge University Press, 1 edn.

MONTRUCCHIO, L., AND F. PRIVILEGGI (2001): "On Fragility of Bubbles in Equilibrium Asset Pricing Models of Lucas-Type," *Journal of Economic Theory*, 101(1), 158–188.

SANTOS, M. S., AND M. WOODFORD (1997): "Rational Asset Pricing Bubbles," *Econometrica*, 65(1), 19–57.