Martingale properties of self-enforcing debt

Florin Bidian * Camelia Bejan[†‡]

February 12, 2012

Abstract

Not-too-tight (NTT) debt limits are endogenous restrictions on debt that prevent agents from defaulting and opting for a specified continuation utility, while allowing for maximal credit expansion (Alvarez and Jermann 2000). For an agent facing some fixed prices for the Arrow securities, we prove that discounted NTT debt limits must differ by a martingale. Discounted debt limits are submartingales/martingales under an interdiction to trade/borrow, and can be supermartingales under a temporary interdiction to trade. With high interest rates and borrowing limited by the agent's ability to repay debt out of his future endowments, nonpositive NTT debt limits are unique. With low interest rates, bubbles limited by the size of the total martingale components in debt limits can be sustained in equilibrium. Bubbles arise in response to debt limits more restrictive (at the prevailing interest rates) than the total amount of self-enforcing debt allowed by the underlying enforcement limitations.

1 Introduction

Alvarez and Jermann (2000) construct a theory of endogenous debt constraints in complete markets economies with limited enforcement of financial contracts. Follow-

*Corresponding author. Robinson College of Business, Georgia State University, PO Box 4036, Atlanta GA 30302-4036. E-mail: fbidian@gsu.edu

†Rice University, MS 22, PO Box 1892, Houston, TX 77251-1892. E-mail: camelia@rice.edu

‡We thank Jan Werner, Beth Allen, Erzo Luttmer, David Rahman and the seminar audiences at University of Minnesota, Georgia State University, Utah State University and the 2011 SAET Conference (Faro, Portugal) for useful comments. All remaining errors are ours.

ing Kehoe and Levine (1993) and Kocherlakota (1996), they assume that agents can default on debt at the cost of being excluded permanently from financial markets. At each date and state, an agent is allowed to borrow the maximum amount which is self-enforcing (making repayment individually rational). These endogenous bounds on debt are referred to as debt limits that are *not-too-tight (NTT)* for the respective agent.

Kocherlakota (2008) uncovered a defining characteristic of the set of NTT debt limits for an agent facing a fixed pricing kernel (or, equivalently, fixed prices of the one-period Arrow securities at each date and state) and penalty for default: adding a martingale to some discounted NTT debt limits results in bounds that are also NTT. The proof is immediate, and is a consequence of agent's budget constraint being unchanged under the martingale-inflated bounds, if the initial value of the martingale is added to his initial wealth.

We prove the converse, which is considerably more involved. A pair of discounted debt limits that are NTT (for a given agent, pricing kernel and penalties for default) *must* differ by a martingale.[1] This theorem does not depend on equilibrium considerations and stems only from the optimizing behavior of the agent. We allow for general penalties for default specified by a continuation utility that can be date and state contingent, and can depend on endogenous variables such as asset prices. When the punishment for default is the *interdiction to borrow*, Hellwig and Lorenzoni (2009a) proved that discounted NTT debt limits are martingales. With this outside option, zero bounds on debt are NTT. Thus their result can be seen as a special case of our theorem.[2]

This characterization of the NTT debt limits (for an agent facing a given pricing kernel and penalties for default) can be used to establish their *uniqueness*, when the present value of agent's endowments is finite, that is with *high interest rates.* In this case, borrowing should be limited by the agent's ability to repay his debt out of his future endowments (Santos and Woodford 1997), or equivalently, by the present value of future endowments. The difference of two such nonpositive discounted NTT debt

[1]Let p and $\bar{\phi}, \phi$ be stochastic processes representing the pricing kernel and two (sequences of) NTT debt limits. Then $p \cdot (\phi - \bar{\phi})$ is a martingale.

[2]Indeed, set $\bar{\phi}$ identically equal to zero at all dates and states. Hence $\bar{\phi}$ is NTT, and the debt limits ϕ are NTT if and only $p \cdot \phi$ $(= p(\phi - \bar{\phi}))$ is a martingale.

limits is therefore a uniformly integrable martingale converging to zero, and hence identically equal to zero. When the punishment for default is the *interdiction to trade*, Alvarez and Jermann (2000, Proposition 4.11) prove that nonpositive NTT debt limits bounded by the present value of debt must exist. Our result establishes that such debt limits are in fact unique. With an interdiction to borrow as punishment for default, debt limits identically equal to zero are NTT, hence uniqueness implies that debt is unsustainable in the presence of high interest rates. This confirms the conclusion reached earlier by Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009a).

The assumption of high interest rates is ad-hoc and extremely restrictive in models with limited enforcement. In these environments, *low interest rates* (making the present value of aggregate endowment infinite) arise in equilibrium as a way to induce agents not to renege on their debt. Such examples are provided in Hellwig and Lorenzoni (2009a) for penalties resulting in an interdiction to borrow, and by Antinolfi, Azariadis, and Bullard (2007) for an interdiction to trade. An adaptation of the theorems of Santos and Woodford (1997) to economies with nonpositive debt constraints as we have here (see Bidian 2011, Chapter 2), rather than borrowing constraints, shows that low interest rates are necessary for the existence of asset price bubbles. The martingale property of NTT debt limits suggests a strong connection to bubbles, as they grow on average at the same rate as the interest rates and therefore they are positive martingales when discounted by the pricing kernel. By not discarding low interest rates on a priori grounds, we are able to pursue this connection.

Kocherlakota (2008) shows that an arbitrary bubble can be injected in the price of an infinitely-lived asset, without altering agents' consumption. This can be accomplished by an upward adjustment of agents' debt limits proportional to the size of the bubble and their initial endowment of the asset, which leaves them NTT. The introduction of a bubble gives consumers a windfall proportional to their initial holding of the asset, which can be sterilized, leaving their budgets unaffected, by an appropriate tightening of the debt limits. He refers to this result as the "bubble equivalence theorem". While an intriguing way to generate bubbles, it raises the question whether the tighter debt bounds needed to sustain the bubble can remain *nonpositive*, due to the bubble component they now contain. Clearly arbitrary large bubble injections can only be sustained by forcing agents to save arbitrary large amounts. Moreover, with high interest rates, even initially infinitesimal bubbles explode quickly and make agents's debt limits positive. Therefore it is unclear whether bubble injections can occur at all with nonpositive debt limits. Positive debt limits force agents to save and seem unreasonable given the presence of enforcement limitations.

We impose nonpositivity of debt limits as an equilibrium requirement. We show that the necessary and sufficient condition for an equilibrium to sustain bubbles is the existence of (negative) martingale components in agents' discounted debt limits. A bubble of size equal to the total martingale component in agents' debt bounds can be injected in equilibrium. Thus the amount of self-enforcing debt restricts the size of a potential bubble. Rational bubbles enable agents to circumvent tight debt limits and to achieve identical allocations to those possible under more relaxed, but still self-enforcing debt limits. Our characterization of NTT debt limits (Theorem 3.5) implies that low interest rates and asset price bubbles must occur in any equilibrium with debt limits that are tighter than maximal self-enforcing levels of debt at the prevailing interest rates.

The type of penalty for default determines the shape of debt limits and the existence of martingale components in them. When agents are allowed to borrow predetermined fixed fractions (possibly zero) of their endowments following default, an equilibrium can sustain bubbles whenever the equilibrium did sustain debt amounts in excess of the penalty levels, since by our theorem, the difference between the equilibrium and the penalty (discounted) debt limits is a martingale. In particular, for the interdiction to borrow case, agents' discounted debt limits are *martingales*, and an equilibrium can sustain bubbles in assets in unit supply equal to the total amount of self-enforcing debt (agents' total debt limits). When the punishment for default is the interdiction to trade, we prove that the discounted NTT debt limits of each agent are *submartingales*, and therefore bubbles can be sustained whenever total amount of self-enforcing debt does not vanish in present value terms. A bubble of initial size equal to the limit of discounted total debt limits can be sustained (for an asset in unit supply).

We present an example in which we describe the equilibria under three types of penalties for default. With an interdiction to trade, respectively borrow, total discounted debt is a submartingale with nonzero limit, respectively a nonzero martingale. With a *temporary (one-period) interdiction to trade*, discounted debt limits are *supermartingales*, containing martingales components. Therefore bubbles can be sustained under all three types of punishments for default. They are a robust and intrinsic feature of economies where restrictions on debt arise endogenously from enforcement limitations. The example illustrates that there is a complex interaction between the severity of the punishment for default, interest rates, the amount of risk sharing and the shape of endogenous debt limits. The amount of equilibrium risk-sharing is not necessarily comonotonic with the (initial) size of sustainable bubbles.

The type of bubbles shown to exist in this paper develop in response to artificially tight credit restrictions at given interest rates (compared to what the underlying enforcement limitations allow). They are a mechanism to preserve the same amount of risk-sharing as afforded by maximal self-enforcing debt limits (at the existing interest rates). Therefore bubbles in this framework are associated to inefficiencies only insofar as the enforcement limitations induce inefficient levels of interest rates and risk sharing in the absence of bubbles. Our example suggests that low interest rates equilibria that can sustain bubbles are not necessarily (constrained) inefficient. They are inefficient for a permanent or temporary interdiction to trade, even though this might be just a byproduct of the stationarity of agents' endowments, as pointed out by Bloise and Reichlin (2011).[3] An interdiction to borrow can lead to both efficient and inefficient equilibria that can sustain bubbles.

The empirical testing and calibration of models with limited enforcement focused solely on constrained efficient equilibria with high interest rates, by following the lead set in foundational theoretical work by Kehoe and Levine (1993, 2001) and Alvarez and Jermann (2000). Alvarez and Jermann (2001) argue that these models deliver too much risk sharing since the resulting pricing kernels are still not volatile enough to explain the equity premium puzzle. Krueger and Perri (2006) also find excessive risk sharing, which can only partially account for the rise in consumption inequality in US. In our example, the equilibria with low interest rates result in less risk sharing than the equilibrium with high interest rates, and by supporting bubbles, they can

[3]They construct an example, similar to ours, but with nonstationary endowments, where an efficient allocation is supported as an equilibrium with low interest rates (under a permanent interdiction to trade after default).

also explain a variety of asset pricing puzzles (Bejan and Bidian 2012). Therefore, allowing for low interest rates has the potential to improve the risk sharing and asset pricing implications of models with limited enforcement. Testing directly for the presence of low interest rates is empirically challenging, and a discussion of the literature is given by Hellwig and Lorenzoni (2009a). Indirectly, one can test for the presence of bubbles in asset prices, which can arise only under low interest rates.

The paper is organized as follows. Section 2 introduces the model, and defines the notion of an Alvarez-Jermann equilibrium, which is a sequential equilibrium where agents are subject to NTT debt limits. In Section 3 we prove that discounted NTT bounds (for a given agent, pricing kernel and penalties for default) are determined only up to a martingale. In Section 4 we give necessary and sufficient conditions for an AJ-equilibrium to sustain bubbles, and show that an interdiction to trade/borrow results in discounted NTT debt limits that are submartingales/martingales. Section 5 contains an example, in which equilibria that can sustain bubbles are constructed for the case where the penalty for default is the permanent or temporary (one-period) interdiction to trade, or the interdiction to borrow. Appendices A and B contain omitted proofs in Section 3 and 5. Appendix C discusses the efficiency of the equilibria in Section 5. The Supplemental Material (Bidian and Bejan 2012) contains three parts. The first part establishes necessary and sufficient transversality conditions for an agent's optimization problem. They are extensions to stochastic environments of the conditions given by Kocherlakota (1992), or alternatively, extensions to nonzero debt constraints of the corresponding conditions in Forno and Montrucchio (2003). The second part presents an elementary proof of Theorem 3.5 for the case when debt constraints bind in bounded time, that requires no martingale techniques or boundedness assumptions on the discounted debt limits. The third part complements results in Section 5.1, showing that all the equilibria that can sustain bubbles under an interdiction to trade can be achieved from fixed, zero initial wealth for the agents.

2 The model

We consider a stochastic, discrete-time, infinite horizon economy. The time periods are indexed by the set of natural numbers $\mathbb{N} := \{0, 1, \ldots\}$. The uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ and by the filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$, which is an increasing sequence of σ-algebras on the set of states of the world Ω generating $\mathcal{F}$, that is such that $\mathcal{F} = \sigma(\cup_t \mathcal{F}_t)$. Each σ-algebra $\mathcal{F}_t$ is interpreted as the information available at period t and is finite. There is no initial information, therefore $\mathcal{F}_0 = \{\emptyset, \Omega\}$. For $\omega \in \Omega$ and $t \in \mathbb{N}$, the set of states that are known to be possible at t if the true state is ω is $\mathcal{F}_t(\omega) := \cap\{A \in \mathcal{F}_t \mid \omega \in A\}$, and is assumed to have positive probability.[4]

A sequence $x = (x_t)_{t\in\mathbb{N}}$ of random variables ($\mathcal{F}$-measurable real-valued functions) is an *adapted stochastic process* ("process" henceforth) if for each $t \in \mathbb{N}$, x_t is $\mathcal{F}_t$-measurable.[5] We let X be the set of all stochastic processes, and denote by X_+ the processes $x \in X$ such that $x_t \geq 0$ P-almost surely ("a.s." henceforth) for all $t \in \mathbb{N}$. We write $x \geq 0$ if x is a nonnegative process, and $x = 0$ if $x_t = 0$ P-a.s. for all $t \in \mathbb{N}$. We write $x \neq 0$ if there exists t such that $x_t = 0$ does not hold (that is, x_t differs from zero on a set of positive probability). All statements, equalities, and inequalities involving random variables are assumed to hold only P-a.s., and we omit this qualifier in what follows. When $K, L \in \mathbb{N} \setminus \{0\}$, let $X^{K\times L}$ be the set of vector (or matrix) processes $(x^{ij})_{1\leq i\leq K, 1\leq j\leq L}$ with $x^{ij} \in X$. For $x \in X^{K\times L}$, we write $x \geq 0$ (respectively $x > 0, x = 0$) if for all $1 \leq i \leq K, 1 \leq j \leq L$ and $t \in \mathbb{N}$, $x_t^{ij} \geq 0$ (respectively $x_t^{ij} > 0, x_t^{ij} = 0$). We write $x \neq 0$ if there exist t, i, j such that $x_t^{i,j} = 0$ does not hold (that is, $x_t^{i,j}$ differs from zero on a set of positive probability). Similarly $x \gneq 0$ means that $x \geq 0$ but $x \neq 0$. The set of nonnegative processes $x \in X^{K\times L}$ (that is, such that $x \geq 0$) is denoted by $X_+^{K\times L}$.

A function $T : \Omega \to \mathbb{N} \cup \{\infty\}$ such that $\{T = n\} \in \mathcal{F}_n$, for all $n \in \mathbb{N}$, is called a *stopping time*. The stopping time T is said to be *finite* if $T < \infty$, and *bounded* if there exists $n \in \mathbb{N}$ such that $T < n$. A stopping time T induces the σ-algebra $\mathcal{F}_T$ of events known at T,

$$\mathcal{F}_T := \{A \in \mathcal{F} \mid A \cap \{T = n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}.$$

[4] Using the usual "event tree" terminology, $\mathcal{F}_t(\omega)$ is the date t node containing state ("leaf") ω (for the parallel between the stochastic processes and event tree language, see Leroy and Werner 2001, chapter 21).

[5] Notice that the process x is *integrable*, since for any $t \in \mathbb{N}$, x_t belongs to the space of integrable random variables $L^1 := L^1(\Omega, \mathcal{F}, P)$, as $\mathcal{F}_t$ is finite.

The operator $E_T(\cdot)$ denotes the conditional expectation with respect to $\mathcal{F}_T$. Let $x = (x_n) \in X$ and T a finite stopping time. The random variable x_T is defined as $x_T(\omega) := x_{T(\omega)}(\omega)$, for all $\omega \in \Omega$. The process x *starting at* T is defined as the sequence of random variables $(x_{T+n})_{n=0}^{\infty}$, which we denote also by $\Theta^T x$ (hence Θ is the familiar shift operator). By extension, if $A \subset X$, then $\Theta^T A := \{\Theta^T x \mid x \in A\}$. Let S be another stopping time, not necessarily finite, such that $T \leq S$. The process x *stopped at* S and starting at T is defined as the sequence of random variables $(x_{(T+n)\wedge S})_{n=0}^{\infty}$, where $(T+n) \wedge S$ is an abbreviated notation for $\min\{T+n, S\}$. We use also the alternative notation $(x_n)_{n=T}^{S}$ for the process x stopped at S and starting at T.

There is a single consumption good and a finite number, I, of consumers. An agent $i \in \{1, 2, \ldots, I\}$ has endowments $e^i \in X_+$, and his preferences are represented by a utility $U^i : X_+ \to \mathbb{R}$ given by $U^i(c) = E \sum_{t=0}^{\infty} u_t^i(c_t)$, where $u_t^i(\cdot) = \beta_t^i u^i(\cdot)$ and $E(\cdot)$ is the expectation operator with respect to the probability P. We assume that $\beta^i \in X_+$ and satisfies $E \sum_{t \geq 0} \beta_t^i < \infty$, and that $u^i : \mathbb{R}_+ \to \mathbb{R}$ is continuous, increasing, strictly concave and bounded from above by $\bar{u}^i \in \mathbb{R}$ and from below by $\underline{u}^i \in \mathbb{R}$. The conditional expectation given the information available at t, $\mathcal{F}_t$, is denoted by $E_t(\cdot)$. Given the absence of information at period 0, $E_0(\cdot) = E(\cdot)$. Let $U_t^i(c) := E_t \sum_{s \geq t} u_s^i(c_s)$ be the continuation utility of agent i after t provided by a consumption stream $c \in X_+$.

Each consumer can trade at each date and state a complete set of one-period Arrow securities. Their prices determine uniquely the pricing kernel $p \in X_{++}$, and conversely, the pricing kernel p determines unambiguously the prices of the Arrow securities.[6] Additionally, there is a finite number J of infinitely-lived, disposable securities. Asset $j \in \{1, 2, \ldots, J\}$ pays dividends $d^j \in X_+$, and has an ex-dividend price per share $q^j \in X_+$. The dividend and price vector processes are $d := (d^1, \ldots, d^J) \in X_+^{1\times J}$ and $q := (q^1, \ldots, q^J) \in X_+^{1\times J}$. Consumer i has an initial endowment $\theta_{-1}^i \in \mathbb{R}_+^J$ of the infinitely-lived securities, and $a_0^i \in \mathbb{R}$ additional wealth, and his trading strategy in the J securities is represented by a process $\theta^i \in X^{J\times 1}$, while his trading strategy in the Arrow securities is given by $a \in X$.

Consumer i faces debt constraints requiring his beginning of period financial

[6] The price at date $t-1$ and state $\omega \in \Omega$ of the Arrow security paying one unit of consumption at t in states $\mathcal{F}_t(\omega)$ is related to the pricing kernel p by the formula $\frac{p_t(\omega)}{p_{t-1}(\omega)} \cdot \frac{P(\mathcal{F}_t(\omega))}{P(\mathcal{F}_{t-1}(\omega))}$.

wealth to exceed some bounds $\phi^i \in X$, meant to prevent Ponzi schemes. Thus if consumer i starts at a finite stopping time T with wealth ν_T ($\mathcal{F}_T$-measurable) and faces constraints ϕ^i and prices p, q, he solves the problem $\max_{(c,a,\theta)\in B^i_T(\nu_T,\phi^i,p,q)} U^i_T(c)$, denoted $P^i_T(\nu_T, \phi^i, p, q)$, where $B^i_T(\nu_T, \phi, p, q)$ is his budget constraint following T, defined as

$$\begin{aligned} B^i_T(\nu_T,&\phi^i, p, q) := \{(c, a, \theta) \in \Theta^T X_+ \times \Theta^{T+1} X \times \Theta^T X^{J\times 1} \mid \\ &c_T + E_T \frac{p_{T+1}}{p_T} a_{T+1} + q_T\theta_T \le e^i_T + \nu_T, \quad a_s + (q_s + d_s)\theta_{s-1} \ge \phi^i_s, \\ &c_s + E_s \frac{p_{s+1}}{p_s} a_{s+1} + q_s\theta_s \le e^i_s + a_s + (q_s + d_s)\theta_{s-1}, \forall s > T\}. \end{aligned} \tag{2.1}$$

The indirect utility of the agent is given by

$$V^i_T(\nu_T, \phi^i, p, q) := \max_{(c,a,\theta)\in B^i_T(\nu_T,\phi^i,p,q)} U^i_T(c). \tag{2.2}$$

Consumer i can elect to default on his debt and receive a continuation utility described by a process $V^{i,d}$. Thus by defaulting at period t, agent i can guarantee for himself a continuation utility $V^{i,d}_t$ (which is $\mathcal{F}_t$-measurable) and can depend on exogenous variables such as agents' endowments, but also on prices p, q, and even future debt limits $\phi^i_{t+1}, \phi^i_{t+2}, \ldots$. When we need to emphasize the functional dependence of penalties on prices and debt limits we use the full notation $V^{i,d}(p, q, \phi^i)$, but in most instances we drop the arguments and do not make the dependence explicit. The debt constraints ϕ^i are determined endogenously to reflect the maximal amount of debt agents can hold without defaulting. We say that the debt limits ϕ^i are *self-enforcing* for agent i at prices p, q given penalties $V^{i,d}$ if $B_t(\phi_t, \phi, p, q) \neq \emptyset$ for all $t \in \mathbb{N}$ and the agent prefers *not* to default, $V^i_t(\phi_t, \phi, p, q) \ge V^{i,d}_t, \forall t \in \mathbb{N}$. The debt limits ϕ^i are *not-too-tight (NTT)* for agent i (at prices p, q) given penalties $V^{i,d}$ if and only if

$$V^i_t(\phi_t, \phi, p, q) = V^{i,d}_t, \forall t \in \mathbb{N}. \tag{2.3}$$

Thus NTT debt limits are self-enforcing bounds that do not restrict credit unnecessarily. Alvarez and Jermann (2000), building on the work of Kehoe and Levine

(1993), assume that the agents are banned from trading following default, that is

$$V_t^{i,d} := U_t^i(e^i), \forall t \in \mathbb{N}. \tag{2.4}$$

Hellwig and Lorenzoni (2009a), following Bulow and Rogoff (1989), allow agents to continue to lend, but not to borrow, upon default. Hence agents can renege on their debt and be required to hold nonnegative wealth thereafter, resulting in a continuation utility that depends on prices,

$$V_t^{i,d} := V_t^i(0, 0, p, q), \forall t \in \mathbb{N}, \tag{2.5}$$

where the second argument in $V_t(0, 0, p, q)$ denotes the process equal to zero at any date and state.

A vector $\left(p, q, (c^i)_{i=1}^I, (a^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\phi^i)_{i=1}^I, (V^{i,d})_{i=1}^I\right)$ consisting of a pricing kernel p, prices q for the infinitely-lived securities, consumption (c^i), trading strategies (a^i) (in Arrow securities) and (θ^i) (in the infinitely-lived securities), debt constraints (ϕ^i) and penalties for default $(V^{i,d})$ is an *AJ-equilibrium* with initial securities holdings $(\theta_{-1}^i)_{i=1}^I$ and initial additional wealth $(a_0^i)_{i=1}^I$ if

i. Consumption and portfolios of each agent i are feasible and optimal: $(c^i, a^i, \theta^i) \in B_0^i(a_0^i + (q_0 + d_0)\theta_{-1}^i, \phi^i, p, q)$ and $U(c^i) = V_0^i(a_0^i + (q_0 + d_0)\theta_{-1}^i, \phi^i, p, q)$.

ii. Markets clear: $\sum_{i=1}^I c_t^i = \sum_{i=1}^I e_t^i, \sum_{i=1}^I \theta_t^i = \sum_{i=1}^I \theta_{-1}^i, \sum_{i=1}^I a_t^i = 0, \forall t \geq 0$.

iii. For each i, ϕ^i is NTT given $V^{i,d}$: $V_t^i(\phi_t^i, \phi^i, p, q) = V_t^{i,d}$, for all $t \geq 0$.

A pricing kernel p and security prices q under which the problem of an agent admits a solution have to exclude arbitrage opportunities, which implies that (see for example Bidian 2011, Chapter 2)

$$q_t = E_t \frac{p_{t+1}}{p_t}(q_{t+1} + d_{t+1}), \forall t \geq 0. \tag{2.6}$$

Therefore $q_t = \frac{1}{p_t} E_t \sum_{s>t} p_s d_s + \lim_{n\to\infty} \frac{1}{p_t} E_t p_n q_n$. Let $f_t(p, d) := \frac{1}{p_t} E_t \sum_{s>t} p_s d_s$ denote the discounted present value at t of future dividends d, that is the *fundamental* *value* of d at period t. It follows that

$$b_t(p,q) := \frac{1}{p_t} \lim_{n\to\infty} E_t p_n q_n \tag{2.7}$$

is well-defined and nonnegative, and $q_t = f_t(p,d) + b_t(p,q)$. The process $b(p,q)$ represents the part of asset prices in excess of fundamental values, and represents the *bubble* component in the asset prices q. Notice that for all $t \in \mathbb{N}$, $p_t b_t(p,q) = E_t p_{t+1} b_{t+1}(p,q)$. Hence $p \cdot b(p,q)$ is a nonnegative martingale,[7] and $b(p,q) = 0$ if and only if $0 = b_0(p,q)$ $(= \frac{1}{p_0} \lim_{t\to\infty} E p_t q_t)$.

3 Characterization of not-too-tight debt limits

There is an intimate connection between NTT debt limits and martingales, which will be explored here. Throughout this section we fix an agent i facing a given pricing kernel p, prices q for the infinitely-lived securities, and penalties for default $V^{i,d}$. We assume that prices p, q exclude arbitrage opportunities, that is they satisfy (2.6). Thus the optimizing agent is concerned only with his total wealth at the beginning of each period, rather than the composition of wealth (infinitely-lived securities versus Arrow securities). If $(c^i, a^i, \theta^i) \in B^i_T(\nu_T, \phi^i, p, q)$, then $(c, a') \in B^i_T(\nu_T, \phi^i, p)$, where for all $s > T$, $a'_s := a_s + (q_s + d_s)\theta^i_{s-1}$ (that is, a'_s is the beginning of period s wealth of the agent), and

$$\begin{aligned} B^i_T(\nu_T, \phi^i, p) :=& \{(c,a) \in \Theta^T X_+ \times \Theta^T X \,|a_T = \nu_T, \\ & c_{T+t} + E_{T+t}\frac{p_{T+t+1}}{p_{T+t}} a_{T+t+1} \le e^i_{T+t} + a_{T+t},\ a_{T+t+1} \ge \phi^i_{T+t+1}, \forall t \ge 0\}. \end{aligned} \tag{3.1}$$

Therefore we focus here on the simpler budgets of the form (3.1), in which we can imagine that the agent is choosing directly the (beginning of period) wealth holdings. We denote the problem $\max_{(c,a)\in B^i_T(\nu_T,\phi^i,p)} U^i_T(c)$ by $P^i_t(\nu_T, \phi^i, p)$, the optimal solution to the problem $P^i_T(\nu_T, \phi^i, p)$ is denoted by $C^i_T(\nu_T, \phi^i, p)$, and the maximum

[7] A process $m \in X$ is a *martingale* if $m_t = E_t m_{t+1}$, for all $t \ge 0$, while m is a *submartingale* (respectively *supermartingale*) if $m_t \le E_t m_{t+1}$ (respectively $m_t \ge E_t m_{t+1}$) for all $t \ge 0$.

continuation utility attainable by the agent is $V^i_T(\nu_T, \phi^i, p)$, that is

$$C^i_T(\nu_T, \phi^i, p) := \text{argmax}_{(c,a)\in B^i_T(\nu_T,\phi^i,p)} U^i_T(c), \tag{3.2}$$

$$V^i_T(\nu_T, \phi^i, p) := \max_{(c,a)\in B^i_T(\nu_T,\phi^i,p)} U^i_T(c). \tag{3.3}$$

As a consequence of the equivalence of the budgets $B^i_T(\nu_T, \phi^i, p, q)$ and $B^i_T(\nu_T, \phi^i, p)$ (from the point of view of consumption), the consumption component in $C^i_T(\nu_T, \phi^i, p, q)$ and $C^i_T(\nu_T, \phi^i, p)$ coincide, and

$$V^i_T(\nu_T, \phi^i, p, q) = V^i_T(\nu_T, \phi^i, p). \tag{3.4}$$

We henceforth drop the last argument (q) in the indirect utility of the agent, as arbitrage opportunities are absent in an equilibrium.

We drop also the agent-specific superscript i for the rest of the section, since we focus on a single agent. We assume that $\bar{\phi} \in X$ are some NTT bounds (for the chosen agent, at prices p and penalties V^d), and that $\phi \in X$ are some alternative debt limits, satisfying $B_t(\phi_t, \phi, p) \neq \emptyset$, for all t. Some of the results of this section require the following assumption on $\bar{\phi}, \phi$:

$$V^d(p, q, \phi) = V^d(p, q, \bar{\phi}), \tag{3.5}$$

that is, continuation utilities after default are the same under the two debt limits. Condition 3.5 is clearly satisfied for penalties such as (2.4) and (2.5) since they do not depend on agent's debt limits. Set

$$M := p(\phi - \bar{\phi}). \tag{3.6}$$

We show next that discounted NTT constraints are determined only up to a martingale, that is we prove that ϕ are NTT (for the given agent at prices p and penalties V^d) if and only if M is a martingale. The "if" part (sufficiency) is immediate, and was shown by Kocherlakota (2008) (for less general penalties for default).

Proposition 3.1. *If M is a martingale, then $V_t(\bar{\phi}_t, \bar{\phi}, p) = V_t(\phi_t, \phi, p)$ for all $t \in \mathbb{N}$ and therefore ϕ are NTT if* (3.5) *holds.*

Proof. It is immediate to check that $(c, a) \in B_t(\bar{\phi}_t, \bar{\phi}, p)$ if and only if $(c, a + \phi - \bar{\phi}) \in B_t(\phi_t, \phi, p)$. Thus for all $t \in \mathbb{N}$, $V_t(\bar{\phi}_t, \bar{\phi}, p) = V_t(\phi_t, \phi, p) = V^d(\phi)$, and equal also to $V^d(\bar{\phi})$ under the additional assumption $V^d(p, q, \phi) = V^d(p, q, \bar{\phi})$, thus $\bar{\phi}$ is NTT. $\square$

The next result is related.

Proposition 3.2. *If M is a supermartingale, then for any $t \geq 0$, $V_t(\phi_t, \phi, p) \geq V_t(\bar{\phi}_t, \bar{\phi}, p)$ with strict inequality on the set $\{M_t > E_t M_{t+1}\}$.*

Proof. It is immediate to check that if $(c, a) \in B_t(\bar{\phi}_t, \bar{\phi}, p)$, then $(\tilde{c}, a + \phi - \bar{\phi}) \in B_t(\phi_t, \phi, p)$, where $\tilde{c}_s := c_s + E_s (M_s - M_{s+1}) / p_s \geq c_s$, for all $s \geq t$. Since $\tilde{c}_t > c_t$ on $\{M_t > E_t M_{t+1}\}$, the conclusion follows. $\square$

Hellwig and Lorenzoni (2009a) proved that the converse of Proposition 3.1 holds, for the particular case (2.5) when agents are not allowed to borrow following default. They prove that if ϕ are NTT, then $p \cdot \phi$ is a martingale. Since in their framework $\bar{\phi} := 0$ are NTT, their result states that $p(\phi - \bar{\phi})(= M)$ is a martingale. We prove that this is the case, for general penalties V^d satisfying (3.5).

Let T be an arbitrary stopping time. Define $\alpha(T)$ to be the first time the bounds ϕ bind after T, when the agent starts with wealth ϕ_T at T and faces bounds ϕ. Concretely, for each $\omega \in \{T < \infty\}$,

$$\alpha(T)(\omega) := \inf \{t \mid t \in \mathbb{N}, t > T(\omega), a_t(\omega) = \phi_t(\omega), (c, a) \in C_T(\phi_T, \phi, p)\}, \qquad (3.7)$$

and for $\omega \in \{T = \infty\}$, $\alpha(T)(\omega) := \infty$. Notice that $\alpha(T)$ is well-defined, as the set $C_T(\phi_T, \phi, p)$ contains a unique element. Indeed, strict concavity of the period utilities (u_t) imply that if if $(c, a), (c', a') \in C_T(\phi_T, \phi, p)$, then $c = c'$, otherwise $((c + c')/2, (a + a')/2) \in B_T(\phi_T, \phi, p)$ would be strictly preferred by the agent to both (c, a) and (c', a'). But then for any $s \geq T$, $V_s(a_s, \phi_s, p) = U_s(c) = U_s(c') = V_s(a'_s, \phi, p)$, hence $a_s = a'_s$ (V_s is strictly increasing), and therefore $(c, a) = (c', a')$. With multiple optimal paths (without strict concavity), our arguments would go through, but we would have to be explicit about which optimal path is selected in the definition of $\alpha(T)$. We also set $\alpha^0(T) := T$ and for $k \geq 1$, we define $\alpha^k(T)$ recursively as $\alpha^k(T) := \alpha(\alpha^{k-1}(T))$.

We present next two ancillary results. First, optimal asset holdings of an agent are nondecreasing in initial wealth. Secondly, for any period t, the process M stopped at $\alpha(t)$ converges almost surely. Proofs are in Appendix A.

Lemma 3.3. *Given any $t \in \mathbb{N}$ and $\mathcal{F}_t$-measurable random variables $\nu' \geq \nu$,*

$$(c,a) \in C_t(\nu,\phi,p), (c',a') \in C_t(\nu',\phi,p) \quad \Rightarrow \quad a'_s \geq a_s, \forall s \geq t. \tag{3.8}$$

Proposition 3.4. *Let $t \in \mathbb{N}$. For $n \in \mathbb{N}$, let $\eta_n := \alpha(t) \wedge n$. If $\sup_{n\geq t} E_t(p_{\eta_n}\phi_{\eta_n})^+ < \infty$,*[8] *then $(p_{\eta_n}\phi_{\eta_n})_{n\in\mathbb{N}}$ converges a.s. If, additionally, $\sup_{n\geq t} E_t\left(p_{\eta_n}\bar{\phi}_{\eta_n}\right)^- < \infty$, then $(M_{\eta_n})_{n\in\mathbb{N}}$ converges a.s.*

We impose the following boundedness conditions on $\phi, \bar{\phi}$.

Assumption 3.1. *For each $t \in \mathbb{N}$, $\sup_{n\geq t} E_t(p_{\eta_n}\phi_{\eta_n})^+ < \infty$, $\sup_{n\geq t} E_t(p_{\eta_n}\bar{\phi}_{\eta_n})^- < \infty$, where $\eta_n := \alpha(t) \wedge n$, and $(M_s)_{s=t}^{\alpha(t)}$ is uniformly integrable and has a uniformly integrable lower Snell envelope.*

The lower Snell envelope of a uniformly integrable process represents the largest submartingale less or equal to the process. While Assumption 3.1 seems hard to verify, it always holds under the following (stronger) condition:

Assumption 3.2. *For each $t \in \mathbb{N}$, the processes $(p\cdot\phi)_{s=t}^{\alpha(t)}$ and $(p\cdot\bar{\phi})_{s=t}^{\alpha(t)}$ are dominated by an integrable random variable, that is $\sup_{n\geq t} |p_{\eta_n}\phi_{\eta_n}| \in L^1$, $\sup_{n\geq t} |p_{\eta_n}\bar{\phi}_{\eta_n}| \in L^1$, where $\eta_n := \alpha(t) \wedge n$.*

Indeed, Assumption 3.2 guarantees that $\inf_{n\geq t} M_{\alpha(t)\wedge n} > -\infty$, and therefore the process $(M_s)_{s=t}^{\alpha(t)}$ has a lower Snell envelope (Kopp 1984, Theorem 2.11.3). Moreover the uniform integrability conditions required by Assumption 3.1 are satisfied under the stronger dominance conditions in Assumption 3.2. The technical boundedness conditions reflected in the above assumptions are needed to insure that the order of limits and expectations can be exchanged. They are very week, since they are imposed piecewise on time intervals $(t, \alpha(t)+1)$, rather than on the whole horizon.

[8] For $x \in \mathbb{R}$, x^+ and x^- denote the positive and negative part of x, $x^+ := -(-x \wedge 0)$ and $x^- := -(x \wedge 0)$.

Therefore if debt limits bind in bounded time, Assumption 3.2 is automatically satisfied. In Section 5.3, we construct an equilibrium[9] in which each agent's discounted debt limits $p \cdot \phi$ are supermartingales converging to $-\infty$, and therefore $p \cdot \phi$ is not dominated by an integrable random variable. However, in that example $\alpha(t) \leq t+2$ for all t, therefore Assumption 3.2 is trivially satisfied.

We can prove now (the harder) converse to Proposition 3.1, which completes the characterization of NTT debt limits.

Theorem 3.5. *If $\bar{\phi}, \phi$ are NTT (given p, q, V^d) and* (3.5) *and Assumption 3.1 hold, then the process $M := p(\phi - \bar{\phi})$ is a martingale.*

Proof. Fix a natural number t.

STEP 1. We show that

$$M_t \geq E_t M_{\alpha(t)}, \tag{3.9}$$

where $M_{\alpha(t)} := \lim_{n\to\infty} M_{\alpha(t)\wedge n}$, which is well-defined by Proposition 3.4. To this end, let $(\hat{M}_s)_{s=t}^{\alpha(t)}$ be the (lower) Snell envelope of $(M_s)_{s=t}^{\alpha(t)}$, that is $\hat{M}_s := \inf_{s \leq T < \alpha(t)+1} E_s M_T$, for $t \leq s < \alpha(t)$ (Kopp 1984, Theorem 2.11.3).[10] It is the largest submartingale dominated from above by M (that is $\hat{M} \leq M$), and it satisfies $\hat{M}_s = M_s \wedge E_s \hat{M}_{s+1}$ and $E_t(\hat{M}_s) = \inf_{s \leq T < \alpha(t)+1} E_t M_T$. Hence there exist a sequence of stopping times (T_n) such that $T_n \nearrow \alpha(t)$ and $\hat{M}_t = \lim_{n\to\infty} E_t M_{T_n} = E_t \lim_{n\to\infty} M_{T_n} = E_t M_{\alpha(t)}$. Moreover, since $(\hat{M}_s)_{s=t}^{\alpha(t)}$ is a uniformly integrable submartingale smaller than M, $\hat{M}_{\alpha(t)} := \lim_{n\to\infty} \hat{M}_{\alpha(t)\wedge n}$ exists, $\hat{M}_{\alpha(t)} \leq M_{\alpha(t)}$ and $\hat{M}_t \leq E_t \hat{M}_{\alpha(t)}$. We conclude that

$$\hat{M}_{\alpha(t)} = M_{\alpha(t)}. \tag{3.10}$$

We prove that $(\hat{M}_s)_{s=t}^{\alpha(t)}$ is in fact a martingale, rather than just a submartingale. Assume, by contradiction, that there exists $n \in \mathbb{N}$ such that $\{t \leq n < \alpha(t)\} \cap \{\hat{M}_n < E_n \hat{M}_{n+1}\}$ has positive probability. Until we reach a contradiction, all statements below are restricted to the set $\{t \leq n < \alpha(t)\} \cap \{\hat{M}_n < E_n \hat{M}_{n+1}\}$ (which is $\mathcal{F}_n$-measurable). Notice that $\hat{M}_n = M_n$, since $\hat{M}_n = M_n \wedge E_n \hat{M}_{n+1}$ and $\hat{M}_n < E_n \hat{M}_{n+1}$.

[9] The penalty for default is a one-period interdiction to trade, see (5.2).

[10] The infimum in the definition of $\hat{M}_s$ refers to the *essential infimum* over all finite stopping times T greater than s and smaller or equal to $\alpha(t)$ (Kopp 1984, Proposition 2.11.1).

Let $(c,a) \in C_n(\phi_n, \phi, p)$. Define $(\tilde{a}_s)_{s=n+1}^{\alpha(t)}$ by

$$\tilde{a}_s := a_s - \frac{\hat{M}_s}{p_s} \geq \phi_s - \frac{M_s}{p_s} = \phi_s - (\phi_s - \bar{\phi}_s) = \bar{\phi}_s,$$

and let $\tilde{a}_n = \bar{\phi}_n$. Let $(\tilde{c}_s)_{s=n}^{\alpha(t)-1}$ be the consumption supported by asset holdings $\tilde{a}$, thus $p_s(\tilde{c}_s - c_s) = p_s(\tilde{a}_s - a_s) - E_s p_{s+1}(\tilde{a}_{s+1} - a_{s+1})$. Hence

$$\begin{aligned} p_n(\tilde{c}_n - c_n) &= -M_n + E_n\hat{M}_{n+1} = -\hat{M}_n + E_n\hat{M}_{n+1} > 0, \\ p_s(\tilde{c}_s - c_s) &= -\hat{M}_s + E_s\hat{M}_{s+1} \geq 0, \quad n+1 \leq s < \alpha(t). \end{aligned}$$

We reached a contradiction, since

$$\begin{aligned} V_n^d = V_n(\bar{\phi}_n, \bar{\phi}, p) &\geq E_n\left(\sum_{s=n}^{\alpha(t)-1} u_s(\tilde{c}_s) + V_{\alpha(t)}^d \mathbf{1}_{\alpha(t)<\infty}\right) \\ > E_n\left(\sum_{s=n}^{\alpha(t)-1} u_s(c_s) + V_{\alpha(t)}^d \mathbf{1}_{\alpha(t)<\infty}\right) &= V_n(\phi_n, \phi, p) = V_n^d. \end{aligned}$$

Having established that $\hat{M}$ is a martingale, (3.9) follows now from (3.10).

STEP 2. We show that

$$M_t = E_t M_{\alpha(t)}. \tag{3.11}$$

For each $k \in \mathbb{N}$, repeat the construction in STEP 1 for $\alpha^k(t)$ instead of t, on the set where $\{\alpha^k(t) < \infty\}$, and obtain the martingale $(\hat{M}_s)_{s=\alpha^k(t)+1}^{\alpha^{k+1}(t)}$ (the lower Snell envelope of $(M_s)_{s=\alpha^k(t)+1}^{\alpha^{k+1}(t)}$), dominated from above by M and such that $\hat{M}_{\alpha^{k+1}(t)} = M_{\alpha^{k+1}(t)}$.[11] We let also $\hat{M}_t := E_t\hat{M}_{\alpha(t)}$. By (3.9), the resulting process $(\hat{M}_s)_{s=t}^{\infty}$ is a supermartingale, $\hat{M} \leq M$, and for all $k \geq 1$, $\hat{M}_{\alpha^k(t)} = M_{\alpha^k(t)}$.

Construct the process $(\hat{\phi})_{s=t}^{\infty}$ defined by $\hat{\phi}_s := \bar{\phi}_s + \hat{M}_s/p_s$. It follows that $\phi_s \geq \hat{\phi}_s$ for all $s \geq t$, and $\hat{\phi}_{\alpha^k(t)} = \phi_{\alpha^k(t)}$ for all $k \geq 1$. Let $(\bar{c}, \bar{a}) \in C_t(\phi_t, \phi, p)$. Then $(\bar{c}, \bar{a})$ satisfies the Kuhn-Tucker and necessary transversality conditions (Bidian and

[11] Equivalently, for $\alpha^k(t) + 1 \leq s < \alpha^{k+1}(t) + 1$ define $\hat{M}_s := E_s M_{\alpha^{k+1}(t)}$ and use repeatedly the property (3.8) and (3.9) to show that $\hat{M} \leq M$.

Bejan 2012, Lemma 1.1)

$$\frac{u'_s(\bar{c}_s)}{u'_{s+1}(\bar{c}_{s+1})} - \frac{p_s}{p_{s+1}} \geq 0, \left(\frac{u'_s(\bar{c}_s)}{u'_{s+1}(\bar{c}_{s+1})} - \frac{p_s}{p_{s+1}}\right)(\bar{a}_{s+1} - \phi_{s+1}) = 0, \forall s \geq t, \tag{3.12}$$

$$\lim_{T\to\infty} E_t \beta^T u'(\bar{c}_T)(\bar{a}_T - \phi_T) = 0. \tag{3.13}$$

We claim that $(\bar{c}, \bar{a})$ is an optimal solution for the problem $P_t(\phi_t, \hat{\phi}, p)$ with relaxed debt limits, that is we show that $(\bar{c}, \bar{a}) \in C_t(\phi_t, \hat{\phi}, p)$. Since $\bar{a}$ binds at the same dates and states under the ϕ and $\hat{\phi}$ bounds, it follows that $(\bar{c}, \bar{a})$ satisfies the Kuhn-Tucker conditions for the problem $P_t(\phi_t, \hat{\phi}, p)$. Let $(c, a) \in B_t(\phi_t, \hat{\phi}, p)$ and $\eta_n := \alpha^k(t) \wedge n$, for $k \geq 1$ and $n \geq t$. By (3.12) and Lemma 1.2 in Bidian and Bejan (2012),

$$\begin{aligned} E_t \sum_{s=t}^{\eta_n - 1} (u_s(c_s) - u_s(\bar{c}_s)) &\leq E_t u'_{\eta_n}(\bar{c}_{\eta_n})(\bar{a}_{\eta_n} - \hat{\phi}_{\eta_n}) \leq \\ &\leq E_t u'_{\eta_n}(\bar{c}_{\eta_n})(\bar{a}_{\eta_n} - \phi_{\eta_n}) + \frac{p_t}{u'_t(\bar{c}_t)} E_t p_{\eta_n}(\phi_{\eta_n} - \hat{\phi}_{\eta_n}) = \\ &= E_t u'_n(\bar{c}_n)(\bar{a}_n - \phi_n)\mathbf{1}_{n \leq \alpha^k(t)} + \frac{p_t}{u'_t(\bar{c}_t)} E_t(M_{\eta_n} - \hat{M}_{\eta_n}), \end{aligned} \tag{3.14}$$

as $(\bar{a}_{\eta_n} - \hat{\phi}_{\eta_n})\mathbf{1}_{n > \alpha^k(t)} = 0$. Using (3.13),

$$\lim_{n\to\infty} E_t u'_n(\bar{c}_n)(\bar{a}_n - \phi_n)\mathbf{1}_{n \leq \alpha^k(t)} \leq \lim_{n\to\infty} E_t u'_n(\bar{c}_n)(\bar{a}_n - \phi_n) = 0,$$

and $\lim_{n\to\infty} E_t(M_{\eta_n} - \hat{M}_{\eta_n}) = 0$ since $\hat{M}_{\alpha^k(t)} = M_{\alpha^k(t)}$ and $\hat{M}^{\alpha^k(t)}, M^{\alpha^k(t)}$ are uniformly integrable. Making $n \to \infty$ in (3.14), $E_t \sum_{s=t}^{\alpha^k(t)-1} (u_s(c_s) - u_s(\bar{c}_s)) \leq 0$. Letting $k \to \infty$, we conclude that $(\bar{c}, \bar{a}) \in C_t(\phi_t, \hat{\phi}, p)$. Therefore $V_t(\phi_t, \hat{\phi}, p) = V_t(\phi_t, \phi, p) = V_t^d$, and

$$V_t^d = V_t(\phi_t, \hat{\phi}, p) \geq V_t(\hat{\phi}_t, \phi', p) \geq V_t^d.$$

The first inequality above is strict if $\phi_t > \hat{\phi}_t$ and the second one is strict if $\hat{M}$ is not a martingale, but rather only a supermartingale, by Proposition 3.2. Thus $\hat{M}$ is a martingale and $\phi_t = \hat{\phi}_t$. Thus $M_t = \hat{M}_t = E_t M_{\alpha(t)}$, and (3.11) obtains.

STEP 3. We show that

$$M_t = E_t M_{t+1}. \tag{3.15}$$

It is enough to prove that

$$M_{t+1} = E_{t+1} M_{\alpha(t)}, \tag{3.16}$$

since then $M_t = E_t M_{\alpha(t)} = E_t E_{t+1} M_{\alpha(t)} = E_t M_{t+1}$, as desired. Let $\eta^0 := t+1$ and for $m \geq 1$, $\eta^{m+1} := \alpha(\eta^m) \wedge \alpha(t)$. Thus $\eta^m \nearrow \alpha(t)$. Fix $l \in \mathbb{N}$. We show first that $M_{\eta^l} = E_{\eta^l} M_{\eta^{l+1}}$. On the set $\{\eta^l < \alpha(t)\}$, the monotonicity property (3.8) implies that $\alpha(\eta^l) \leq \alpha(t)$, thus $\eta^{l+1} = \alpha(\eta^l)$. By (3.11),

$$\mathbf{1}_{\eta^l < \alpha(t)} \cdot E_{\eta^l} M_{\eta^{l+1}} = \mathbf{1}_{\eta^l < \alpha(t)} \cdot E_{\eta^l} M_{\alpha(\eta^l)} = \mathbf{1}_{\eta^l < \alpha(t)} \cdot M_{\eta^l}.$$

On the set $\{\eta^l = \alpha(t)\}$, $\eta^{l+1} = \eta^l = \alpha(t)$. Therefore

$$E_{\eta^l} M_{\eta^{l+1}} = \mathbf{1}_{\eta^l < \alpha(t)} \cdot E_{\eta^l} M_{\eta^{l+1}} + \mathbf{1}_{\eta^l = \alpha(t)} \cdot E_{\eta^l} M_{\eta^l} = \mathbf{1}_{\eta^l < \alpha(t)} \cdot M_{\eta^l} + \mathbf{1}_{\eta^l = \alpha(t)} \cdot M_{\eta^l} = M_{\eta^l}.$$

Using the law of iterated expectations, it follows that

$$M_{t+1} = M_{\eta^0} = E_{\eta^0} M_{\eta^1} = \ldots = E_{\eta^0} M_{\eta^l} = E_{t+1} M_{\eta^l}, \forall l \in \mathbb{N}.$$

By the dominated convergence theorem,

$$M_{t+1} = \lim_{l \to \infty} E_{t+1} M_{\eta^l} = E_{t+1} \lim_{l \to \infty} M_{\eta^l} = E_{t+1} M_{\alpha(t)}.$$

Therefore (3.16) holds and hence (3.15) is true, thus M is a martingale. $\square$

The idea of the proof is depicted in Figure 1. In Step 1 we construct the Snell envelope of M on the interval $[t, \alpha(t)]$ (the largest submartingale smaller than M), and show that it has to be in fact a martingale (otherwise the agent will default when faced with debt limits ϕ). It follows that the process M *sampled* at t and $\alpha(t)$ is a supermartingale. In Step 2, we construct in a similar fashion the Snell Envelope $\hat{M}$ for the process M on the intervals $[t, \alpha(t)], (\alpha(t), \alpha^2(t)], (\alpha^2(t), \alpha^3(t)], \ldots$. By Step 1, $\hat{M}$ is a supermartingale. Using $\hat{M}$, we construct the relaxed bounds $\hat{\phi} := \bar{\phi} + \hat{M}/p \leq \phi$, which coincide with ϕ at $\alpha(t), \alpha^2(t), \ldots$, that is whenever ϕ are binding in the problem $P_t(\phi_t, \phi, p)$. Therefore the optimal solution for $P_t(\phi_t, \phi, p)$ is also a solution of the

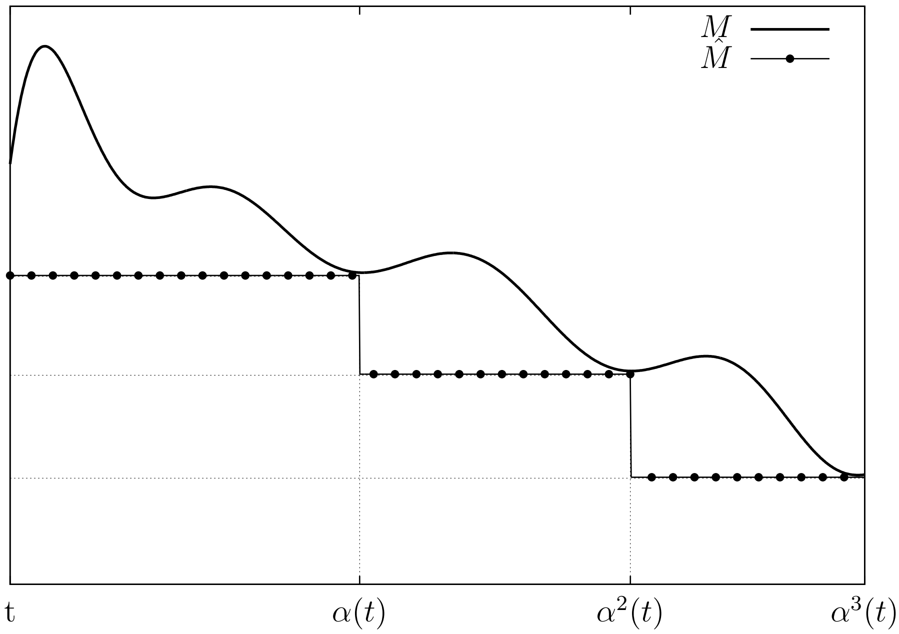

Figure 1: Illustration of the proof of Theorem 3.5.

relaxed problem (with larger feasible set) $P_t(\phi_t, \hat{\phi}, p)$ if it satisfies the transversality condition for the relaxed problem. We show that this is indeed the case, and by Proposition 3.2, we conclude that $\phi_t = \hat{\phi}_t$, and therefore the process M sampled at t and $\alpha(t)$ is a martingale (rather than just a supermartingale, as shown in Step 1). Since t was arbitrary, Step 3 uses an optional sampling type of argument in reverse, and proves that M must be a martingale.

An immediate consequence of Theorem 3.5 is the uniqueness of nonpositive[12] NTT debt limits that are bounded by the present value of agent's future endowments, assumed finite.

Proposition 3.6. *For each* $t \in \mathbb{N}$, *let* $Y_t := \frac{1}{p_t} E_t \sum_{s \geq t} p_s e_s$ *and assume* $Y_0 < \infty$. *Let* $\phi, \bar{\phi}$ *be NTT given* V^d *and satisfying* (3.5). *If* $0 \geq \phi, \bar{\phi} \geq -Y$, *then* $\phi = \bar{\phi}$.

Proof. Notice that the process $p \cdot Y$ is a uniformly integrable positive supermartingale converging to zero a.s. and in L^1. Thus Assumption 3.1 is satisfied with $\alpha(t)$ replaced by ∞, and the conclusion follows by Theorem 3.5.

[12]Nonpositivity implies that there is no forced saving.

In fact, because of the strong boundedness and convergence assumptions implicit here, we can give also a direct proof for this proposition using only the first part of the proof of Theorem 3.5. The processes $p \cdot \phi$, $p \cdot \bar{\phi}$, $M := p \cdot (\phi - \bar{\phi})$ and $M' := p \cdot (\bar{\phi} - \phi) = -M$ are bounded (in absolute value) from above by $p \cdot Y$, and therefore they converge to zero a.s. and in L^1 (being uniformly integrable). Moreover the lower Snell envelopes of the processes M, M' exist, since M and M' are bounded from below by the uniformly integrable submartingale $-p \cdot Y$. Repeating the argument in STEP 1 of Theorem 3.5, with $\alpha(t)$ replaced by ∞, it follows that for all $t \geq 0$, $M_t \geq E_t \lim_{n \to \infty} M_n = 0$. In the same manner, with M' taking the place of M, we infer that $M' \geq 0$. Thus $M = 0$ and hence $\phi = \bar{\phi}$. $\square$

Therefore with *high interest rates* and borrowing limited by the agent's ability to repay his debt out of his future endowments (Santos and Woodford 1997), nonpositive NTT debt limits are unique (for a given agent, pricing kernel and penalties for default). Proposition 3.6 fills some gaps and gives a unified view of results obtained for various penalties for default. When the punishment for default is the interdiction to trade, Alvarez and Jermann (2000, Proposition 4.11) prove that given any sequential equilibrium with NTT debt limits and high interest rates, one can construct an equivalent equilibrium with identical pricing kernel and consumption, but with nonpositive NTT debt limits bounded by the present value of aggregate endowment. Proposition 3.6 shows that such debt limits are in fact unique. Moreover, when the punishment for default is the loss of borrowing privileges, nonpositive NTT debt limits restricted by the present value of future endowments must be identically equal to zero, and therefore no borrowing can be sustained in an equilibrium, as pointed out before by Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009a).

The assumption of high interest rates is ad-hoc and extremely restrictive in models with limited enforcement. In these environments, low interest rates arise in equilibrium as a way to induce agents not to default. In fact, in the next section, we show that Theorem 3.5 implies that low interest rates and asset price bubbles must occur in any equilibrium with debt limits that are tighter than maximal self-enforcing levels of credit at the prevailing interest rates. Rational bubbles enable agents to circumvent tight debt limits and to achieve identical allocations to those possible under more relaxed, but still self-enforcing debt limits.

4 Bubble injections

The martingale characterization of NTT debts in Proposition 3.1 and Theorem 3.5 can be used to show that robust bubbles can arise in limited enforcement economies. The self-enforcing debt that can be sustained in an equilibrium can be converted into asset price bubbles. We compare pairs of AJ-equilibria, therefore to avoid lengthy notation, we set $\mathcal{E} := \big(p, q, (c^i)_{i=1}^I, (a^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\phi^i)_{i=1}^I, (V^{i,d})_{i=1}^I\big)$, while $\bar{\mathcal{E}}$, $\widetilde{\mathcal{E}}$, $\hat{\mathcal{E}}$ denote similar vectors, with all variables barred, tilded, respectively hatted. We say that the AJ-equilibria $\mathcal{E}, \hat{\mathcal{E}}$ are *equivalent* if pricing kernels, consumptions and penalties for default coincide: $\hat{p} = p$, $\hat{c}^i = c^i$, $\hat{V}^{i,d} = V^{i,d}$, for all agents i.

Kocherlakota (2008) showed that an arbitrary bubble can be injected in an infinitely-lived asset, while leaving agents' budget constraints (hence consumption) unchanged, as long as the debt constraints of the agents are allowed to be adjusted upwards by their initial endowment of the asset multiplied by the bubble term. The introduction of a bubble gives consumers a windfall proportional to their initial holding of the asset, which can be sterilized, leaving their budgets unaffected, by an appropriate tightening of the debt limits. He refers to this result as "the bubble equivalence theorem". The modified debt constraints bind in exactly the same dates and states, and they are NTT if the initial bounds were NTT (in his paper, penalties for default do not depend on asset prices or agents' debt limits). The result goes through in our framework with more general penalties for default, assuming that they are not affected by the the addition of a bubble. Given a pricing kernel p, we let $M(p)$ be the set of nonnegative processes that are martingales when discounted by p, $M(p) := \{\varepsilon \in X_+ | p \cdot \varepsilon \text{ martingale }\}$. Similarly, $M^J(p) := (M(p))^J \subset X_+^{1\times J}$.

Proposition 4.1 (Kocherlakota 2008)**.** *Let $\mathcal{E}$ be an AJ-equilibrium without bubbles and $\varepsilon \in M^J(p)$. Assume that penalties $V^{i,d}$ of each agent i satisfy $V^{i,d}(p, q, \phi^i) = V^{i,d}(p, \hat{q}, \hat{\phi}^i)$, where $\hat{q} = q + \varepsilon$ and $\hat{\phi}^i := \phi^i + \varepsilon \cdot \theta^i_{-1}$. Then $\hat{\mathcal{E}}$ is an equivalent AJ-equilibrium having asset price bubbles given by ε, where for each agent i and each period $t \geq 0$, $\hat{q} = q + \varepsilon$, $\hat{\theta}^i_{t-1} = \theta^i_{t-1}$, $\hat{a}^i_t := a_t + \varepsilon_t(\theta^i_{-1} - \theta^i_{t-1})$ and $\hat{\phi}^i_t := \phi^i_t + \varepsilon_t \cdot \theta^i_{-1}$.*

The proof is immediate and relies on the equality of agents' budgets constraints in $\mathcal{E}$ and $\hat{\mathcal{E}}$. Market clearing conditions are clearly satisfied. Bounds $\hat{\phi}^i$ remain NTT by Proposition 3.1, as $V^i_t(\hat{\phi}^i_t, \hat{\phi}^i, p) = V^i_t(\phi^i_t, \phi^i, p) = V^{i,d}_t(p, q, \phi^i) = V^{i,d}_t(p, \hat{q}, \hat{\phi}^i)$, for all t. An AJ-equilibrium $\mathcal{E}$ satisfies automatically the condition

$$V^{i,d}(p, q, \phi^i) = V^{i,d}(p, q + \varepsilon, \phi^i + \varepsilon \cdot \theta^i_{-1}), \quad \forall \varepsilon \in M^J(p), \forall i \tag{4.1}$$

if the penalties for default do not depend on debt limits and prices (interdiction to trade (2.4)), or if they don't depend on debt limits and depend on prices only through the pricing kernel (interdiction to borrow (2.5)). In Section 5.3 we analyze an example with penalties for default (one-period interdiction to trade, see (5.2)) that depend on debt limits and asset prices, and which nevertheless satisfy (4.1).

Proposition 4.1 uses only the "easy" direction in the characterization of NTT bounds of Section 3 (Proposition 3.1) and it can be applied to more general environments. If the long-lived securities dynamically complete the markets or if markets are incomplete, then a bubble injection can alternatively be achieved through a change in agents' trading of infinitely-lived securities, rather than through an adjustment of agents' holdings of Arrow securities (Bejan and Bidian 2012, Theorem 2.3).

The bubble equivalence theorem did not receive the attention it deserves, since it was universally assumed that the new (tighter) debt bounds ($\hat{\phi}^i$) required to sustain the bubble injection in a positive supply asset must eventually become positive, due to the bubble component they now contain, implying that agents are subjected to forced saving. This is indeed the case in the presence of *high interest rates* that make the present value of aggregate endowment finite. Intuitively, the bubble component added to debt limits explodes and makes them positive eventually, since it grows on average at the rate of interest. Formally, the claim follows by adapting the results of Santos and Woodford (1997), derived for the case of borrowing constraints, to economies with debt constraints. This is done in Bidian (2011, Chapter 2), where it is shown that no bubbles can exist in assets in positive supply if the present value of aggregate endowment is finite and if agents are subject to nonpositive debt limits.[13] Therefore *low interest rates* (which make the present value of aggregate endowment infinite) are a necessary condition for the tighter debt bounds ($\hat{\phi}^i$) of Proposition 4.1 to remain nonpositive when bubbles are added to assets in positive supply.

[13]Hellwig and Lorenzoni (2009a) make the same point by appealing to the results of Santos and Woodford (1997) on the nonexistence of bubbles. However, it is unclear how to apply Santos and Woodford's (1997) results to their environment with one-period assets, where bubbles are impossible, and where agents are subject to debt rather than borrowing constraints.

We subscribe to the view that debt limits have to be nonpositive, as forced saving seems implausible with enforcement limitations. For the rest of the paper, we *make nonpositivity of debt limits a part of the definition of an AJ-equilibrium.* Given an AJ-equilibrium, we say that it *can sustain bubbles (in assets in positive supply)* if there exists an equivalent equilibrium that has a bubble in one of the assets (in positive supply). Proposition 4.1 seems to suggest that for an equilibrium to sustain bubbles in positive supply assets, it must be the case that the discounted debt limits of *all* agents that have nonzero endowments of the security must contain negative martingale components. This condition is stronger than needed. The next result develops necessary and sufficient conditions under which an equilibrium can sustain bubbles, and characterizes the size of those bubbles. Bubbles can be sustained if and only if at least *one* agent has discounted debt limits having a negative martingale component. The size of the bubbles is limited by the total martingale components in agents' debt limits.

Proposition 4.2. *Let $\mathcal{E}$ be an AJ-equilibrium with nonpositive debt limits and penalties satisfying* (4.1) *in any equivalent equilibrium. Let $\varepsilon, m^1, \ldots, m^I \in M(p)$ such that $\phi^i \leq -m^i$ for each agent i, and $\varepsilon \cdot \sum_{i=1}^I \theta_{-1}^{i,j} = \sum_{i=1}^I m^i$, for some asset $j \in \{1, \ldots, J\}$. If $\varepsilon \neq 0$, there exists an equilibrium equivalent to $\mathcal{E}$, with nonpositive debt limits and in which the price of asset j is $q^j + \varepsilon$, and therefore asset j has a bubble ε. Conversely, if $\mathcal{E}$ has the bubble $\varepsilon \neq 0$ in the price of asset j, then there exists an equilibrium $\hat{\mathcal{E}}$ equivalent to $\mathcal{E}$ such that $\hat{\phi}^1 \leq -\sum_{i=1}^I m^i$.*

Proof. We construct an equilibrium $\bar{\mathcal{E}}$ equivalent to $\mathcal{E}$, with identical debt limits for the agents, in which agent 1 has all the initial endowment of infinitely lived securities. This can be accomplished by setting for all $t \geq 0$, $\bar{\theta}_{t-1}^1 = \sum_{i=1}^I \theta_{-1}^i$, $\bar{\theta}_{t-1}^i = 0$ if $i > 1$, and $\bar{a}_t^i := a_t^i + (q_t + d_t)(\theta_{t-1}^i - \bar{\theta}_{t-1}^i)$ for all i. Showing that $\bar{\mathcal{E}}$ is an AJ-equilibrium is immediate, since agents have identical wealth levels as in the initial equilibrium, and only the distribution of this wealth between Arrow securities and infinitely-lived assets is changed.

For the first part, we construct an equilibrium $\widetilde{\mathcal{E}}$ equivalent to $\bar{\mathcal{E}}$, in which agents' debt limits are $\widetilde{\phi}^1 = \phi^1 + m^1 - \sum_{i=1}^I m^i$, $\widetilde{\phi}^i = \phi^i + m^i$ for $i > 1$, and for all $i \geq 1$, $\widetilde{a}^i = \bar{a}^i + \widetilde{\phi}^i - \bar{\phi}^i$, $\widetilde{\theta}^i = \bar{\theta}^i$. $\widetilde{\mathcal{E}}$ is indeed equivalent to $\bar{\mathcal{E}}$ by the equality of agents' budget constraints established in Proposition 3.1. The conclusion then follows directly from

Proposition 4.1, applied to $\widetilde{\mathcal{E}}$ instead of $\mathcal{E}$.

The second part (the converse) follows from Proposition 4.1, by injecting the discounted martingale $-\varepsilon$ in the price of asset j in the equilibrium $\bar{\mathcal{E}}$. Indeed, price of asset j remains positive after the negative martingale injection due to the existing bubble component, and Proposition 4.1 is valid, as agents' budgets are identical and market clearing conditions hold. It should be remarked that a negative martingale injection in an equilibrium without bubbles would lead to negative prices, unraveling this argument. $\square$

Proposition 4.2 shows that an any AJ-equilibrium can sustain bubbles of arbitrary size on assets in *zero supply*. Moreover, it can sustain bubbles equal to the sum of martingale components in agents' debt limits on assets in *unit supply*.

We investigate next the existence of (negative) martingale components in debt limits for some concrete punishments for default, including the most common penalties used in the literature (see (2.4)) and (2.5)). We consider first a relatively general situation where penalties for default for each agent are described by some exogenous nonpositive debt restrictions $\bar{\phi}^i$ for each agent i. By defaulting at t, the agent has his debt discharged, in exchange for a "fee" $|\bar{\phi}^i_t|$ at t and tighter future debt limits $\bar{\phi}^i$. The penalties $\bar{\phi}^i \leq 0$ can be arbitrarily small in absolute value, or even zero, in which case we have an interdiction to borrow upon default, (2.5). For example, $|\bar{\phi}^i|$ can be taken to be an arbitrary fraction of agent's i income.

Proposition 4.3. *Let $(\bar{\phi}^1, \ldots, \bar{\phi}^I) \in -X_+^{1\times I}$. Consider an AJ-equilibrium $\mathcal{E}$ with debt limits $\phi^i \leq \bar{\phi}^i \leq 0$, and penalties for default given by $V_t^{i,d} := V_t^i(\bar{\phi}^i_t, \bar{\phi}, p, q)$, for all i, t. If $\phi^i, \bar{\phi}^i$ and p satisfy Assumption 3.1 for each i, then $\mathcal{E}$ can sustain bubbles in assets in positive supply if and only if $\sum_{i=1}^I(\phi^i - \bar{\phi}^i) \neq 0$. Any $\varepsilon \in M(p)$ satisfying $\varepsilon \sum_{i=1}^I \theta_{-1}^{i,j} = \sum_{i=1}^I(\phi^i - \bar{\phi}^i)$ can be injected in asset j as a bubble.*

Proof. Theorem 3.5 ensures that for each agent i, the discounted debt limits $p \cdot (\phi^i - \bar{\phi}^i)$ are negative martingales. The conclusion follows from Proposition 4.2, since the penalties for default here do not vary with debt limits and depend on prices only through the pricing kernel, hence (4.1) holds. $\square$

Proposition 4.3 showcases the full power of Theorem 3.5. It shows that under penalties for default described by some exogenous nonpositive debt limits $(\bar{\phi}^i)$, an equilibrium can sustain bubbles whenever the equilibrium did sustain debt levels in excess of the penalty levels. In particular, for the interdiction to borrow case (2.5), agents' discounted debt limits are *martingales*, and an equilibrium can sustain bubbles in assets in positive supply if and only if at least one agent is allowed to borrow (that is, he is subject to nonzero and nonpositive debt limits).

We analyze now the case when the punishment for default is the interdiction to trade (2.4). We show that discounted NTT debt limits are *submartingales*, and therefore bubble injections resulting in nonpositive debt constraints are possible if and only if at least one agent has discounted debt bounds with nonzero limit (non-vanishing discounted debt limits).

Theorem 4.4. *Let $\mathcal{E}$ be an AJ-equilibrium with penalties* (2.4) *(no trading after default). Then for each agent i, $p \cdot \phi^i$ is a submartingale converging a.s. Bubbles in positive supply assets can be sustained if and only $\lim_{t\to\infty} p_t \sum_{i=1}^{I} \phi_t^i \neq 0$. Any $\varepsilon \in M(p)$ satisfying $\varepsilon_n \sum_{i=1}^{I} \theta_{-1}^{i,j} = \frac{1}{p_n} \lim_{t\to\infty} p_t \sum_{i=1}^{I} \phi_t^i$ for all $n \in \mathbb{N}$ can be injected in asset j as a bubble.*

Proof. Fix an agent i and a period t. Agent i will default at period t, when starting with wealth ϕ_t^i at period t, on the set $\{p_t \phi_t^i > E_t p_{t+1} \phi_{t+1}^i\}$. Indeed, let $(c, a) \in C_t^i(\phi_t^i, \phi^i, p)$. Construct $(c', a') \in B_t^i(\phi_t^i, \phi^i, p)$ (see (3.1)) given by $c'_t := e_t^i + \left(p_t \phi_t^i - E_t p_{t+1} \phi_{t+1}^i\right)/p_t$, $a'_t := \phi_t^i$, and $(c', a') \in C_{t+1}^i(\phi_{t+1}^i, \phi^i, p)$ (hence $a'_{t+1} := \phi_{t+1}^i$). On the set $\{p_t \phi_t^i > E_t p_{t+1} \phi_{t+1}^i\}$, $c'_t > e_t^i$, and

$$U_t^i(c') = u_t^i(c'_t) + E_t V_{t+1}^{i,d} > u_t^i(e_t^i) + E_t V_{t+1}^{i,d} = u_t^i(e_t^i) + E_t U_{t+1}^i(e^i) = V_t^{i,d}.$$

It follows that $U_t^i(c') > U_t^i(c) = V_t^{i,d}$ on the set $\{p_t \phi_t^i > E_t p_{t+1} \phi_{t+1}^i\}$, contradicting the optimality of the path c. Hence $p_t \phi_t^i \leq E_t p_{t+1} \phi_{t+1}^i$ for all t and therefore $p \cdot \phi$ is a submartingale. Since $\phi \leq 0$, the martingale convergence theorem (Kopp 1984, Theorem 2.6.1) applies, and $(p_t \phi_t^i)$ converges a.s. to an integrable variable.

For each agent i, let $Z^i := \lim_{t\to\infty} p_t \phi_t^i \quad (\leq 0)$. A simple argument based on Fatou's lemma shows that $p_t \phi_t^i \leq E_t Z^i$ (Kopp 1984, Remark 2.6.5). Define $m_t^i := -E_t Z^i / p_t$, for all t and i. By construction, $m^i \in M(p)$ and $\phi^i \leq -m^i$. Moreover, $\sum_{i=1}^{I} Z^i \neq 0$ if and only if $\sum_{i=1}^{I} m^i \neq 0$. The conclusion follows from Proposition 4.2. $\square$

In the next section we introduce a deterministic monetary economy and study the AJ-equilibria under three types of penalties for default, showing that bubbles can be sustained in equilibrium.

5 An example

We consider a deterministic economy with two agents $\{e, o\}$ with endowments alternating between a high and a low value, but with constant aggregate endowment, as in Woodford (1990), Kocherlakota (1992, Example 1) or Huang and Werner (2000, Example 7.1). However, here we introduce enforcement limitations.

Agent e (o) has high endowment y^H at even (odd) periods, and low endowment y^L at odd (even) periods, with $y^L < y^H$. Thus for all $t \geq 0$, $e^e_{2t} = e^o_{2t+1} = y^H$ and $e^e_{2t+1} = e^o_{2t} = y^L$. Agent $i \in \{e, o\}$ faces debt bounds (ϕ^i_t) and has a utility $U^i(c) := \sum_{t \geq 0} u_t(c_t)$, where $u_t(c_t) = \beta^t u(c_t)$, with $\beta \in (0, 1)$ and u is strictly increasing, strictly concave and twice differentiable. We assume that there is enough heterogeneity in agents' income or that the discount rate β is high enough so that interest rates at an autarchic equilibrium are low,

$$\beta u'(y^L)/u'(y^H) > 1. \tag{5.1}$$

The only infinitely-lived asset is fiat money, paying zero dividends and assumed in unit supply. Each agent $i \in \{e, o\}$ has an initial nonegative endowment of money $\theta^i_{-1} \geq 0$, and additional wealth (in the form of Arrow securities) a^i_0. We consider (alternatively) three types of penalties for default $V^{i,d}$: interdiction to trade (2.4), interdiction to borrow (2.5), and a temporary, one-period interdiction to trade following default, after which agents are granted a "fresh-start" and receive back their initial endowment of money,

$$V^{i,d}_t := u_t(y^i_t) + V^i_{t+1}(q_{t+1}\theta^i_{-1}, \phi, p), \forall t \in \mathbb{N}. \tag{5.2}$$

We focus on AJ-equilibria $(p, q, (c^i), (a^i), (\phi^i), (V^{i,d}))$ with *unvalued* money, that is with $q = 0$ where, at each period, the agent with low endowment (*low-type* agent) is borrowing constrained, while the high endowment agent (*high-type* agent) is unconstrained. Thus high-type agents always start the period with wealth equal to their debt limit. We call such equilibria *cyclical*, since each agent alternates between being borrowing constrained or not. Using the results of Section 4, we show that (some of) these equilibria can sustain bubbles, thus they are equivalent to equilibria with *valued* money, in which the value of money is limited by the amount of self-enforcing debt existing in the bubble-free equilibrium.

In any cyclical equilibria, the first order conditions for agent $i \in \{e, o\}$ in the problem $P^i(a_0^i, \phi^i, p)$ (at allocations with positive consumption) are

$$u'(c_t^i) \geq \beta u'(c_{t+1}^i)\frac{p_t}{p_{t+1}}, \text{ with “=” if } a_{t+1}^i > \phi_{t+1}^i. \tag{5.3}$$

Denote the endowment minus consumption of the high-type agent at period t by x_t, with $0 \leq x_t < y^H$. Thus x_t represents the *transfer* from the high-type agent to the low-type agent at t. Let $\pi_t := p_{t+1}/p_t$ be the one-period bond price at t (the price at t of the Arrow security paying one unit of good at $t + 1$). Then the first order conditions (5.3) for the two agents are equivalent to

$$\frac{u'(y^H - x_t)}{\beta u'(y^L + x_{t+1})} = \frac{p_t}{p_{t+1}} \quad \left(= \frac{1}{\pi_t}\right), \forall t \geq 0, \tag{5.4}$$

$$\frac{u'(y^L + x_t)}{\beta u'(y^H - x_{t+1})} \geq \frac{u'(y^H - x_t)}{\beta u'(y^L + x_{t+1})}. \tag{5.5}$$

Notice that (5.5) can be written as

$$\frac{u'(y^L + x_t)}{u'(y^H - x_t)} \geq \frac{u'(y^L + (y^H - y^L - x_{t+1}))}{u'(y^H - (y^H - y^L - x_{t+1}))},$$

which holds if and only if

$$x_t + x_{t+1} \leq y^H - y^L. \tag{5.6}$$

Therefore transfers (x_t) and bond prices (π_t) are compatible with agents' first order conditions if and only if (5.4) and (5.6) are satisfied. Given transfers (x_t), pricing kernels and consumptions are

$$p_0 := 1,\ p_{t+1} := \prod_{s=0}^{t} \frac{\beta u'(y^L + x_{s+1})}{u'(y^H - x_s)}; \quad c_t^e := e_t^e - (-1)^t x_t,\ c_t^o := e_t^o + (-1)^t x_t. \tag{5.7}$$

We preview the results to follow. When the penalty for default is the *interdiction to trade*, there exists a stationary equilibrium characterized by constant transfers $\min\{\bar{x}, (y^H - y^L)/2\}$, where $\bar{x}$ is the unique strictly positive number satisfying

$$u(y^H - \bar{x}) + \beta u(y^L + \bar{x}) = u(y^H) + \beta u(y^L). \tag{5.8}$$

Moreover, $\bar{x} < (y^H - y^L)/2$ if and only

$$(1 + \beta)u((y^H + y^L)/2) < u(y^H) + \beta u(y^L), \tag{5.9}$$

in which case there is imperfect risk sharing, otherwise perfect risk sharing characterized by constant transfers $(y^H - y^L)/2$ obtains.[14] In the stationary equilibrium, interest rates are high (bond prices are less than 1), and therefore bubbles cannot be sustained in equilibrium. However, for each initial transfer $0 < x_0 < \min\{\bar{x}, (y^H - y^L)/2\}$, there exists a nonstationary equilibrium (x_t) converging monotonically to autarchy, $x_t \searrow 0$. In all these equilibria, discounted debt limits are *submartingales*, and the total credit in the economy equals the equilibrium transfers, $x_t = -\phi_t$, where $\phi_t := \sum_{i \in \{e,o\}} \phi^i_t$. Moreover $(p_t x_t)$ is a decreasing sequence with a non-zero limit, and therefore a bubble of maximal initial size $\lim_{t\to\infty} p_t x_t$ $(= -\lim_{t\to\infty} p_t \phi_t)$ and vanishing asymptotically (due to low interest rates) can be sustained in all nonstationary equilibria (Proposition 4.2 and Theorem 4.4).

When the penalty for default is the *interdiction to borrow*, there exists a unique stationary equilibrium, characterized by transfers $x^* < \min\{\bar{x}, (y^H - y^L)/2\}$ such that equilibrium interest rates are zero (bond prices are 1),

$$u'(y^H - x^*) = \beta u'(y^L + x^*). \tag{5.10}$$

As in the previous case, for any initial transfer $0 < x_0 < x^*$, there are nonstationary equilibria (x_t) converging monotonically to autarchy, $x_t \searrow 0$. In the stationary and nonstationary equilibria, discounted debt limits are *martingales* and $x_t = -\phi_t$ $(\phi_t := \sum_{i \in \{e,o\}} \phi^i_t)$. All these equilibria can sustain bubbles of initial size x_0 (Propo-

[14] The inequality in (5.9) can be understood as requiring that the first best symmetric allocation in which each agent consumes half of the aggregate endowment does not satisfy the participation constraints of the high type agents.

sition 4.3), constant in the stationary equilibrium (and equal to x^*) but vanishing asymptotically in the nonstationary equilibria.

With the *temporary interdiction to trade* (5.2), we show that there exist stationary equilibria with even less risk sharing, $\hat{x} < x^*$, under parameter conditions where there would be perfect risk sharing under a permanent interdiction to trade. In such an equilibrium, interest rates are low, debt limits for the high-type agents are $\phi^H < 0$ and for the low-type are $\phi^L < 0$, and discounted debt limits are *supermartingales*. Notice that penalties (5.2) satisfy (4.1). Indeed,

$$\begin{aligned} V_t^{i,d}(p, q+\varepsilon, \phi^i + \theta^i_{-1}\varepsilon) &= u_t(e^i_t) + V^i_{t+1}((q_{t+1} + \varepsilon_{t+1})\theta^i_{-1}, \phi^i + \theta^i_{-1}\varepsilon, p) \\ &= u_t(e^i_t) + V^i_{t+1}(q_{t+1}\theta^i_{-1}, \phi^i, p) = V_t^{i,d}(p, q, \phi^i), \end{aligned} \tag{5.11}$$

where the first and last equality follow from the definition of penalties (5.2), while the middle equality holds by Proposition 3.1. Therefore by Proposition 4.2, a bubble of maximal initial size $-(\phi^H + \phi^L)$ and vanishing asymptotically (because of interest rates greater than 1) can be sustained in equilibrium. It can be shown that $-(\phi^H + \phi^L) < \hat{x}$ $(< x^*)$, therefore punishment (5.2) sustains both less risk sharing and smaller initial bubbles than an interdiction to borrow (in stationary equilibria). The interdiction to trade, on the other hand, sustains the maximum amount of risk sharing (perfect risk-sharing for these parameters) and no bubble in a stationary equilibrium. Therefore the equilibrium amount of risk sharing is not necessarily comonotonic to the size of the bubble that can be sustained.

5.1 Interdiction to trade

We analyze first the case when the punishment for default is the interdiction to trade. Alvarez and Jermann (2001) focused only on stationary equilibria with high interest rates in this environment. Antinolfi, Azariadis, and Bullard (2007) pointed out that, with initial transfers between agents, in addition to the stationary cyclical equilibrium, there are an infinite number of nonstationary ones. However, they have not computed the NTT debt limits supporting these allocations, which is crucial for understanding whether bubbles can be sustained. We characterize fully these nonstationary equilibria and show that they can sustain bubbles. This conclusion is robust to fixing agents' initial wealth levels. Indeed, in the supplementary material (Bidian and Bejan 2012, Section 3), we show that all equilibrium paths (both stationary or nonstationary) mentioned above can be reached after a one-period transition, from a zero initial wealth for all agents.[15] The NTT conditions (applied to high-type agents) give

$$u(y^H - x_t) + \beta u(y^L + x_{t+1}) = u(y^H) + \beta u(y^L). \tag{5.12}$$

We construct sequences (x_t) satisfying (5.6) and prices p, trading strategies a^i and bounds ϕ^i supporting the transfers (x_t) as an AJ-equilibrium. Let

$$f(x_t, x_{t+1}) := u(y^H) + \beta u(y^L) - u(y^H - x_t) - \beta u(y^L + x_{t+1}). \tag{5.13}$$

Proposition 5.1. *Let $\bar{x}$ be the unique strictly positive solution of $f(\bar{x}, \bar{x}) = 0$. Choose x_0 such that $0 \leq x_0 \leq \min\left\{\bar{x}, \frac{y^H - y^L}{2}\right\}$. There exists a unique sequence $(x_t)_{t\geq 0}$ satisfying $f(x_t, x_{t+1}) = 0$ for all $t \geq 0$, and $(x_t)_{t\geq 0}$ is strictly decreasing to 0 if $0 < x_0 < \bar{x}$ and constant if $x_0 \in \{0, \bar{x}\}$. Moreover, (x_t) are the transfers from high-type to low-type agents in a cyclical AJ-equilibrium $(p, q, (c^i), (a^i), (\phi^i), (V^{i,d}))$ with unvalued money ($q = 0$), outside options $V^{i,d}$ given by (2.4), and for all $t \geq 0$, p and (c^i) are given by (5.7), while (a^i) and the nonnegative debt limits (ϕ^i) satisfy*

$$-a^o_{t+1} = a^e_{t+1} = \frac{a^e_0 + L(t)}{p_{t+1}},\ \phi^e_{2t} = a^e_{2t},\ \phi^e_{2t+1} = \frac{p_{2t+2}}{p_{2t+1}}\phi^e_{2t+2},\ \phi^o_t = -x_t - \phi^e_t,$$

with $L(t) := \sum_{s=0}^{t}(-1)^s p_s x_s$. Initial wealth levels are $a^e_0 := -a^o_0$ and a^o_0 is arbitrarily chosen in the interval $[L_1, L_2]$, with $L_1 := \lim_{t\to\infty} L(2t-1)$, $L_2 := \lim_{t\to\infty} L(2t)$. Limits L_1, L_2 exist as $(p_t x_t)$ is strictly decreasing if $x_0 > 0$, and $0 \leq L_1 \leq L_2$.

The proof is given in Appendix B. In all the equilibria constructed in Proposition 5.1, the total self-enforcing amount of credit $-(\phi^e + \phi^o)$ equals the transfers between agents x, but the actual allocation of debt limits between agents is indeterminate. This is not surprising, since it is known from proposition 3.1 that martingale components added to debt limits leave agents' budget constraints unchanged if the initial wealth of the agent is increased by the initial value of the martingale. Therefore the

[15]With zero initial wealth, there exists an equilibrium in which the transfers from the high-type to low-type agents are constant after the first period and an infinite number of nonstationary equilibria converging to autarchy.

indeterminacy in debt limits is achieved by varying agents' initial wealth.

In the stationary equilibrium with constant transfers $\min\{\bar{x}, (y^H - y^L)/2\}$, interest rates are high. Indeed, assume first that $\bar{x} < (y^H - y^L)/2$, which happens if and only if $\bar{f}((y^H - y^L)/2) > \bar{f}(\bar{x})(= 0)$ ($\bar{f}(x) = f(x, x)$), or equivalently, if and only if (5.9) holds.[16] Bond prices are constant and equal to some $\pi < 1$ as $\bar{x} > x^*$ with x^* given in (5.10), as shown at the beginning of the proof of Proposition 5.1. If, on the other hand, $\bar{x} \geq (y^H - y^L)/2$, that is if (5.9) is violated, then the constant sequence of transfers equal to $(y^H - y^L)/2$ (in which consumers get half of the aggregate endowment) leads to equilibrium bond prices equal to $\beta < 1$. Thus the present value of aggregate endowment is finite in the stationary case, and fiat money injections would lead to positive debt limits, as argued in Section 4.

The next proposition shows that if agents have *hyperbolic absolute risk aversion (HARA)* utility functions (Leroy and Werner 2001, p.96), the cyclical nonnstationary equilibria associated to transfers (x_t) constructed in Proposition 5.1 can support injections of valued fiat money as in Section 4, while preserving the nonpositivity of the upwardly adjusted debt limits. As shown in Theorem 4.4, the necessary and sufficient condition for such bubble injections is that the discounted debt limits of at least one agent do not vanish asymptotically, that is

$$\lim_{t\to\infty} p_t(\phi_t^e + \phi_t^o) < 0, \text{ or equivalently, } \lim_{t\to\infty} p_t x_t > 0. \tag{5.14}$$

Proposition 5.2. *Assume that agents have HARA utilities. Any nonstationary cyclical equilibrium associated to transfers* (x_t) *with* $0 < x_0 < \min\left\{\bar{x}, \frac{y^H - y^L}{2}\right\}$ *and* $f(x_t, x_{t+1}) = 0$ *for all* $t \geq 0$ *(as described in Proposition 5.1) satisfies* (5.14)*, and therefore can sustain bubble injections.*

Proof. By (5.4) and (B.2),

$$p_{t+1}x_{t+1} = \frac{p_0}{x_0}\prod_{s=0}^{t}\frac{x_{s+1}/x_s}{p_s/p_{s+1}} \geq \frac{p_0}{x_0}\prod_{s=0}^{t}\frac{u'(y^H)/u'(y^H - x_s)}{u'(y^L)/u'(y^L + x_{s+1})}.$$

[16] In order for (5.1) and (5.9) to hold jointly, β must satisfy

$$u'(y^H)/u'(y^L) < \beta < \left(u(y^H) - u((y^H + y^L)/2)\right) / \left(u((y^H + y^L)/2) - u(y^L)\right),$$

and the strict concavity of u guarantees that β belongs to a nonempty interval.

Assume that agents have HARA utilities $u(c) := (\alpha + \gamma c)^{1-\frac{1}{\gamma}} / (\gamma - 1)$ defined on $\{c \,|\, -\alpha < \gamma c\}$. We assume that $\alpha, \gamma \geq 0$, and therefore any positive consumption belongs to the allowed domain. As usual, for $\gamma = 1$, $u(c) := \ln(\alpha + c)$ and for $\gamma = 0$, $u(c) := -e^{-\alpha c}$.

For $\gamma > 0$ (that is, for power or log utilities),

$$\frac{u'(y^H)/u'(y^H - x_s)}{u'(y^L)/u'(y^L + x_{s+1})} = \left(\frac{\left(\alpha + \gamma(y^H - x_s)\right) / \left(\alpha + \gamma y^H\right)}{\left(\alpha + \gamma(y^L + x_{s+1})\right) / \left(\alpha + \gamma y^L\right)} \right)^{\gamma} = \left(\frac{1 - \gamma x_s/(\alpha + \gamma y^H)}{1 + \gamma x_{s+1}/(\alpha + \gamma y^L)} \right)^{\gamma}.$$

As $x_t \searrow 0$, there exists $t_0 \in \mathbb{N}$ such that $x_t \leq \ln 2$ for all $t \geq t_0$. Using the inequalities

$$e^x \geq 1 + x \quad \forall x \in \mathbb{R}, \quad e^{-x} < 1 - x/2 \quad \forall x \in (0, \ln 2],$$

it follows that for all $t \geq t_0$,

$$\frac{p_{t+1} x_{t+1}}{p_{t_0}/x_{t_0}} \geq \prod_{s=t_0}^{t} e^{-\frac{2\gamma x_s}{\alpha + \gamma y^H} - \frac{\gamma x_{s+1}}{\alpha + \gamma y^L}} \geq \prod_{s=t_0}^{t} e^{-\frac{3\gamma x_s}{\alpha + \gamma y^L}} \geq e^{-\frac{3\gamma}{\alpha + \gamma y^L} \sum_{s=t_0}^{\infty} x_s}. \tag{5.15}$$

For $\gamma = 0$ (that is, for exponential utility),

$$\frac{p_{t+1} x_{t+1}}{p_0/x_0} = \prod_{s=0}^{t} e^{-\alpha(x_s + x_{s+1})} \geq \prod_{s=0}^{t} e^{-2\alpha x_s} \geq e^{-2\alpha \sum_{s=0}^{\infty} x_s}. \tag{5.16}$$

Since $x_t \searrow 0$, by (B.2) it follows that there exists $0 < l < 1$ such that $\frac{x_{t+1}}{x_t} < l$ for all t large enough, which implies the convergence of the series $\sum x_t$. Therefore $(p_t x_t)$ is bounded away from zero, hence $\lim p_t x_t > 0$, by (5.15) and (5.16). $\square$

Therefore we showed that for a large class of utility functions the discounted total debt limits do not vanish in the nonstationary AJ-equilibria, and therefore bubbles can be sustained in equilibrium. The HARA utility assumption in Proposition 5.2 simplifies the proof, and it can likely be relaxed.

As mentioned at the beginning of this section, each non-autarchic cyclical equilibrium described in Proposition 5.1 requires specific non-zero initial wealth for the agents. However, in Bidian and Bejan (2012, Section 3), we show that all such cyclical equilibrium paths can be reached after a one-period transition, when all agents start with predetermined, zero wealth.

5.2 Interdiction to borrow

The case where agents are not allowed to borrow after default was discussed also in Hellwig and Lorenzoni (2009b), who show that from any initial level of transfers x_0 less than x^* given by (5.10), there exist a unique sequence of transfers $x_t \searrow 0$ forming an equilibrium, as long as agents' period utility u has a coefficient of risk aversion less than one (on consumption levels below the aggregate endowment). We establish that these results hold also for utilities with coefficients of relative risk aversion higher than one.

The martingale property of NTT bounds guaranteed by Theorem 3.5,[17] that is the fact that $p_t\phi_t^i = p_{t+1}\phi_{t+1}^i$ for all i and t, simplifies the task of characterizing cyclical equilibria. Indeed, consider a cyclical AJ-equilibrium $(p, q, (c^i), (a^i), (\phi^i), (V^{i,d}))$ with $q = 0$ and outside options $V^{i,d}$ given by (2.5). Let $\phi_t := \phi_t^e + \phi_t^o$. If agent i is the high-type at t, his budget constraint gives ($j \in \{e, o\} \setminus \{i\}$ being the low-type at t)

$$x_t = y^H - c_t^i = -\phi_t^i + \frac{p_{t+1}}{p_t} a_{t+1}^i = -\phi_t^i + \frac{p_{t+1}}{p_t}(-\phi_{t+1}^j) = -\phi_t^i - \phi_t^j = -\phi_t. \quad (5.17)$$

It follows that $x_t = -\phi_t$ for all t, as it was the case for the equilibria in Proposition 5.1, (where agents were not allowed to trade after default). Therefore for all $t \geq 0$, $p_t x_t = p_{t+1}x_{t+1}$, or equivalently, $x_{t+1} = x_t/\pi_t$. By (5.4), $h(x_t, \pi_t) = 0$, where

$$h(x, \pi) := \frac{u'(y^H - x)}{\beta u'(y^L + x/\pi)} - \frac{1}{\pi}. \quad (5.18)$$

Proposition 5.3. *Let x^* be given by* (5.10) *(equivalently, x^* is the unique solution of $h(x^*, 1) = 0$). Choose x_0 such that $0 \leq x_0 \leq x^*$. Assume that the utility u has a coefficient of relative risk aversion less than $1 + y^L/x^*$ for consumption in the interval $[y^L, y^L + y^H]$. Then there are unique sequences $(x_t)_{t\geq 0}$ and $(\pi_t)_{t\geq 0}$ such that $h(x_t, \pi_t) = 0$ and $x_{t+1} = x_t/\pi_t$ for all $t \geq 0$. When $x_0 = x^*$ then $x_t = x^*$ and $\pi_t = 1$ for all t. When $x_0 = 0$ then $x_t = 0$ and $\pi_t = \beta u'(y^L)/u'(y^H)$ for all t. When $0 < x_0 < x^*$ then $x_t \searrow 0$ and $\pi_t \nearrow \beta u'(y^L)/u'(y^H)$. The sequences (x_t)*

[17] Assumption 3.1 is satisfied, as debt limits bind in bounded time (in at most 2 periods).

and (π_t) *represent the transfers from high-type to low-type agents and bond prices in a cyclical AJ-equilibrium* $(p, q, (c^i), (a^i), (\phi^i), (V^{i,d}))$ *with unvalued money* $(q = 0)$*, outside options* $V^{i,d}$ *given by* (2.5)*, and for all* $t \geq 0$*,* p *and* (c^i) *are given by* (5.7)*, while* (a^i) *and the nonnegative debt limits* (ϕ^i) *satisfy*

$$a_{2t}^e := -\frac{a_0^o}{p_{2t}}, a_{2t+1}^e := \frac{x_0 - a_0^o}{p_{2t+1}}, a_{t+1}^o := -a_{t+1}^e;\ \phi_t^e := -\frac{a_0^o}{p_t},\ \phi_t^o := -x_t - \phi_t^e.$$

The initial wealth a_0^o *of the odd agent is arbitrarily chosen in the interval* $[0, x_0]$*.*

The proof is given in Appendix B. Proposition 5.3 constructs a class of *nonstationary* cyclical equilibria converging to autarchy. Autarchy is also an equilibrium, characterized by zero transfers $x_t = 0$ for all t. Additionally, there exists a *stationary* equilibrium with transfers $x_t = x^*$ for all t, which allows less risk sharing than the stationary equilibrium of Proposition 5.1 where agents were not allowed to trade after default (characterized by transfers $x_t = \bar{x}$ for all t). Indeed, as shown at the beginning of the proof of Proposition 5.1, $x^* < \bar{x}$. This is in line with the intuition that a more severe punishment for default would facilitate the extension of credit and let agents smooth consumption better. As in the case of an interdiction to trade as punishment for default, in all the equilibria uncovered in Proposition 5.3 the total self-enforcing amount of credit $-(\phi^e + \phi^o)$ that arises endogenously equals the transfers between agents x, but the actual split of debt allowances (limits) between agents depends on their initial wealth, which can take arbitrary values in some interval.

By Proposition 4.3, bubble injections of maximal initial size $-\phi_0 = x_0$ are possible in each of the equilibria of Proposition 5.3. Thus only the autarchic equilibrium cannot sustain bubbles. Notice also that all cyclical AJ-equilibrium allocations described in Proposition 5.3 can be achieved from a zero initial wealth for the agents.

5.3 Temporary interdiction to trade

We relax the complete interdiction from trade as punishment for default, and we assume that the penalty for default is (5.2). In a bubble-free equilibrium (that is, with unvalued money), (5.2) coincides with the penalty analyzed in Azariadis and Kaas (2008), where after default agents are excluded from access to financial markets for one period. In line with their findings, we show that this mild punishment can guarantee steady state partial risk sharing even under parameter conditions where full risk sharing would obtain under a permanent interdiction to trade after default. New to this paper are conditions under which these steady state equilibria allow for even less risk sharing than an interdiction to borrow, and therefore where low interest rates prevail. Moreover, we characterize the NTT debt limits, showing that they are supermartingales on a subsequence (when discounted by the pricing kernel), and that bubbles can be sustained in equilibrium. Under mild additional parametric assumptions, we prove that the discounted debt limits are in fact supermartingales.

As before, we focus on cyclical equilibria with unvalued money. Let $a_t > 0$ be the beginning of period t wealth of a low-type at t. Thus $\pi_t a_{t+1}$ represents the savings of a high type at t. Let also $\phi_t^H := -a^t$ be the debt limit at t of a high-type at t, and ϕ_t^L the debt limit of a low-type. Assume that agent i is the high-type at t. He is indifferent between defaulting or not at t, since $a_t^i = -a_t = \phi_t^i$. The monotonicity property (3.8) implies that debt limits bind at $t+2$ in the problem $P_{t+1}^i(0, \phi^i, p)$ (if the agent i defaults at t), since they are binding in the problem $P_{t+1}^i(a_{t+1}^i, \phi^i, p)$ and $a_{t+1}^i = a_{t+1} > 0$. It follows that

$$u_t(y^H - x_t) + u_{t+1}(y^L + x_{t+1}) + V_{t+2}^{i,d} = u_t(y^H) + u_{t+1}(y^L - \pi_{t+1}\phi_{t+2}^i) + V_{t+2}^{i,d}. \quad (5.19)$$

Either agent's budget constraint at t gives

$$x_t = a_t + \pi_t a_{t+1}. \quad (5.20)$$

With this notation, (5.19) becomes

$$u(y^H - x_t) + \beta u(y^L + x_{t+1}) = u(y^H) + \beta u(y^L + \pi_{t+1} a_{t+2}). \quad (5.21)$$

Any positive sequences of transfers (x_t) and asset holdings (a_t) (for low types in the corresponding period) satisfying the difference equations (5.20)-(5.21) and (5.6), bond prices (π_t) given by (5.4), debt limits $\phi_t^H = -a_t$ for the high-type at t and some $\phi_t^L < 0$ for the low-type at t form an equilibrium, as long as ϕ_t^L satisfies the NTT condition. Indeed, the participation constraints of low types are clearly satisfied, since their beginning of period wealth is positive, and they are subject to identical debt limits upon reentering the market following default.

We focus on non-autarchic stationary equilibria with imperfect risk-sharing, thus we look for solutions $(x_t), (a_t)$ of (5.20) and (5.21) such that $x_t = \hat{x} \in (0, (y^H - y^L)/2)$, $a_t = \hat{a}$ and $\pi_t = \hat{\pi}$, where $\hat{\pi} = \pi(\hat{x}) := \beta u'(y^L + \hat{x})/u'(y^H - \hat{x})$. Debt limits of a high-type are $\phi^H := -\hat{a}$ and of a low type are some $\phi^L < 0$. The transfer $\hat{x}$ must be a zero of the function

$$g(x) := u(y^H) + \beta u\left(y^L + \frac{\pi(x)}{1+\pi(x)}x\right) - u(y^H - x) - \beta u(y^L + x). \tag{5.22}$$

We assume that (x^* is given by (5.10))

$$u(y^H - x^*) + \beta u(y^L + x^*) > u(y^H) + \beta u(y^L + x^*/2). \tag{5.23}$$

Notice that $u(y^H - x^*) + \beta u(y^L + x^*) > (1+\beta)u\big((y^H + y^L)/2\big)$ and

$$u(y^H) + \beta u(y^L + x^*/2) < u(y^H) + \beta u\left(y^L + (y^H - y^L)/4\right).$$

Therefore sufficient conditions for (5.23) consist in a strengthening of (5.1) to

$$\beta > \frac{u(y^H) - u\big((y^H + y^L)/2\big)}{u\left((y^H + y^L)/2\right) - u\left(y^L + (y^H - y^L)/4\right)}, \tag{5.24}$$

and requiring that u is concave enough so that β can be chosen less than 1,

$$u\big((y^H + y^L)/2\big) - u\big(y^L + (y^H - y^L)/4\big) > u(y^H) - u\big((y^H + y^L)/2\big). \tag{5.25}$$

Proposition 5.4. *Assume that* (5.23) *holds (sufficient conditions are* (5.24)*-*(5.25)*). There exists a stationary AJ-equilibrium with transfers $\hat{x} \in (0, x^*)$ such that $g(\hat{x}) = 0$, bond prices $\hat{\pi} := \beta u'(y^L + \hat{x})/u'(y^H - \hat{x}) > 1$, pricing kernel $p_{t+1} := \hat{\pi}^t$ for all $t \geq 0$ (and $p_0 = 1$), and beginning of period asset holdings $-\hat{a} := -\hat{x}/(1+\hat{\pi})$ and debt limits $\phi^H := -\hat{a} < 0$ for the high-type, respectively $\hat{a}$ and $\phi^L < 0$ for the low-type. Moreover, $\phi^L > \hat{\pi}\phi^H$, that is $p_t\phi^L > p_{t+1}\phi^H$, for all t. A sufficient condition for $\phi^H \geq \hat{\pi}\phi^L$ to hold, that is for $p_t\phi^H \geq p_{t+1}\phi^L$ to be true, is*

$$u(y^L - \hat{a}/\hat{\pi} + \hat{\pi}\hat{a}) - u(y^L) \geq \beta u'(y^H - \hat{x})\hat{a}, \tag{5.26}$$

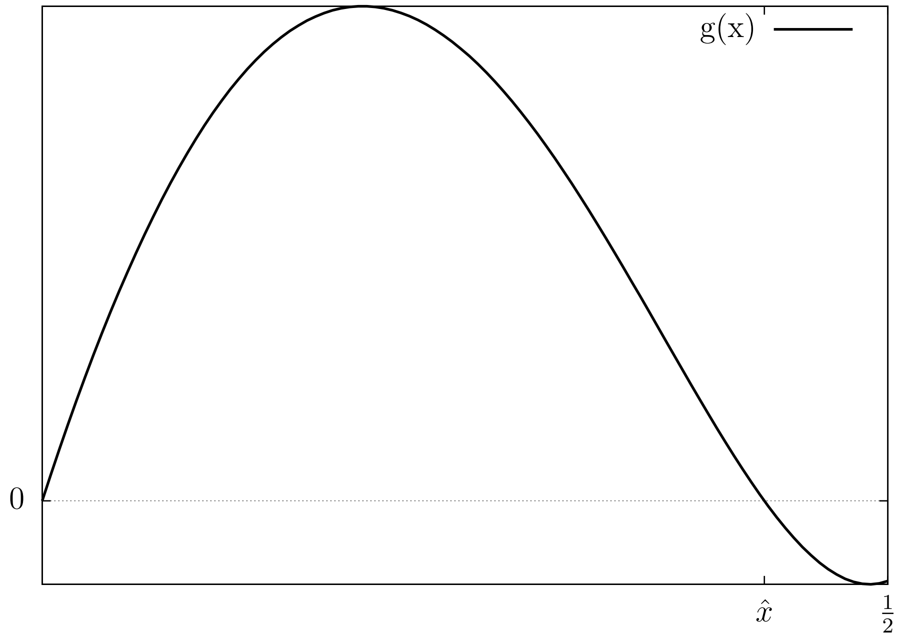

Figure 2: Numerical example: $y^L = 1$, $y^H = 2$, $\beta = 0.99$, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $\gamma = 3$.

which therefore guarantees that agents' discounted debt limits are supermartingales.

The proof is in Appendix B. As $p_t \nearrow \infty$ and $\phi^L, \phi^H > 0$, Proposition 4.3 guarantees that bubbles of initial size $-(\phi^H + \phi^L)$ can be sustained. From $\phi^L > \hat{\pi}\phi^H$ and (5.20),

$$-(\phi^H + \phi^L) < -\phi^H - \hat{\pi}\phi^H = \hat{x} \quad (< x^*),$$

therefore punishment (5.2) sustains both less risk sharing and smaller initial bubbles than an interdiction to borrow (in the stationary equilibrium).

The equilibrium in Proposition 5.4 is described without having a general analytic solution for $\hat{x}$, the zero of $g(x)$ in the interval $(0, x^*)$. To show that there exist parameters that jointly satisfy (5.23) and (5.26), we explore a numerical example.

Example 5.1
As a numerical illustration, let $y^L = 1$, $y^H = 2$, $\beta = 0.99$, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $\gamma = 3$. Function g is represented in Figure 2. It follows that $x^* \approx 0.497$ and the chosen parameters satisfy (5.23). The equilibrium transfers, prices and asset holdings are

$\hat{x} \approx 0.427$, $\hat{\pi} \approx 1.326$ and $\hat{a} \approx 0.184$, thus (5.26) is satisfied. The debt limits of the low-type are obtained from (B.4), by computing $V^i_{t+1}(0, \phi^i, p)$, with agent i being the high-type at $t+1$. By (3.8), agent i is a saver at $t+1$ in the problem $P^i_{t+1}(0, \phi^i, p)$, since he is a saver at $t+1$ in the problem $P^i_{t+1}(\hat{a}, \phi^i, 0)$. Let $(c, a) \in C^i_{t+1}(0, \phi^i, p)$. We guess, and then verify, that agent i is borrowing-constrained at period $t+2$ (when he is low-type) in the problem $P^i_{t+1}(0, \phi^i, p)$. If this is the case, the Euler equation and budgets constraints imply that $u'(c_{t+1})/u'(c_{t+2}) = \beta/\hat{\pi}, \quad c_{t+1} + \hat{\pi}c_{t+2} - \hat{\pi}^2\hat{a} = y^H + \hat{\pi}y^L$. It follows that $c_{t+1} \approx 2 - 0.344$, and $c_{t+2} \approx 1 + 0.503$. We can confirm that our guess was correct, and the agent is indeed borrowing constrained at $t+2$, as $u'(c_{t+2})/u'(y^H - \hat{x}) \geq \beta/\hat{\pi}$. Therefore (B.4) rewrites as

$$u(y^L + \phi^H + \hat{\pi}\hat{a}) + \beta(u(y^H - \hat{x}) + \beta u(y^L + \hat{x})) = u(y^L) + \beta(u(c_{t+1}) + \beta u(c_{t+2})),$$

from which we get $\phi^L \approx -0.197$. The debt limits of the high type are given by $\phi^H = -\hat{a} \approx -0.184$. As $\phi^L > \hat{\pi}\phi^H$ and $\phi^H > \hat{\pi}\phi^L$, agents' debt limits are supermartingales when discounted by the pricing kernel ($p_{t+1} = \hat{\pi}^t$). Therefore the maximum initial size of a bubble that can be sustained in equilibrium is $-\phi^L - \phi^H = 0.381$.

With this parametric values, we contrast the equilibrium here with the equilibria of Propositions 5.1/5.3 for an interdiction to trade/borrow. Notice that (5.9) does not hold, therefore under an interdiction to trade there is perfect risk sharing in a stationary equilibrium, with transfers $\frac{1}{2}$ from high-types to low-types. Interest rates are high and bubbles cannot exist. In a nonstationary equilibrium with an initial value $x_0 = 0.499$ (close to the stationary level of transfers), $\lim_{t\to\infty} p_t x_t \approx 0.244$, which represents also the maximal initial size of a bubble that can be sustained (from the initial level of transfers $x_0 = 0.499$). Thus an interdiction to trade generates (initial) smaller bubbles than a temporary interdiction to trade, which in turn are smaller than under an interdiction to borrow. Indeed, under an interdiction to borrow, maximal initial size of a bubble is $x^* \approx 0.497$ in the stationary equilibrium, or some x_0 (which can be arbitrarily close to x^*) in the nonstationary equilibrium with initial transfers x_0. The amount of (initial) risk sharing however is maximal under an interdiction to trade, followed by an interdiction to borrow, and then by a temporary interdiction to trade. Thus amounts of risk sharing allowed by different penalties are not necessarily comonotonic with the size of bubbles that can be sustained.

6 Conclusion

We build a theory of rational bubbles that jointly predicts their size (limited by the amount of self-enforcing debt in the system), conditions favoring them (unnecessarily tight credit restriction given the underlying contractual and enforcement limitations), and a potential disconnect between the real and financial side of an economy (as the real side is unaffected by bubbles). The setup is an infinite horizon, complete markets economy, in which agents have the option to default on debt at any period in exchange for a continuation utility that can be date and state contingent, and can depend on the pricing kernel.

For an agent facing a given pricing kernel and penalty for default, we characterize the set of debt limits that allow for maximum credit expansion while preventing default, à la Alvarez and Jermann (2000), known as "not-too-tight" (NTT) debt limits. We show that two discounted NTT debt limits for an agent facing a given pricing kernel must differ by a martingale. Our characterization is crucial for showing that the tighter bounds resulting from the injection of a bubble using Kocherlakota's (2008) mechanism can remain nonpositive, despite the bubble component they contain. Indeed, if agents are still allowed to borrow predetermined fixed fractions (arbitrarily small and possibly zero) of their endowments upon default, an equilibrium can sustain bubbles (on assets in unit supply) equal to the total debt limits in excess of the penalty levels. When the punishment for default is the interdiction to borrow, respectively trade, discounted NTT debt limits of each agent are martingales, respectively submartingales, and bubbles of initial size equal with the value, respectively asymptotic value, of total debt limits can be sustained.

We illustrate the sustainability of bubbles in an example in which we compute the equilibria under three types of penalties: permanent or temporary (one-period) interdiction to trade, or interdiction to borrow. The temporary interdiction to trade gives rise to discounted debt limits that are supermartingales. The example reveals that the size of bubbles is not necessarily co-monotonic with the amount of risk sharing that can be sustained in equilibrium, and that equilibria supporting bubbles are not always constrained inefficient.

Thus economies with endogenous (NTT) debt limits provide robust examples of bubbles, in the presence of fully rational, forward looking agents. Bejan and Bidian (2012) point out that bubble injections can occur also with incomplete markets. They also show that bubbles can lead to increases in the volume of trade and can explain a large number of asset pricing puzzles.

A Omitted proofs in Section 3

Proof of Lemma 3.3

Proof. It is enough to prove that $a'_{t+1} \geq a_{t+1}$ and the conclusion follows by iteration. If $c'_t < c_t$, then on $\{a'_{t+1} > \phi_{t+1}\}$ it must be that $a'_{t+1} \leq a_{t+1}$, as V_{t+1} is strictly concave by standard arguments and the first order conditions are (we drop the fixed arguments p, ϕ in the indirect utility function)

$$\frac{u'_t(c'_t)}{V'_{t+1}(a'_{t+1})} = \frac{p_t}{p_{t+1}} \leq \frac{u'_t(c_t)}{V'_{t+1}(a_{t+1})}.$$

Moreover, on $\{a'_{t+1} = \phi_{t+1}\}$, $\phi_{t+1} = a'_{t+1} \leq a_{t+1}$, thus $a'_{t+1} \leq a_{t+1}$. This contradicts $a_t \leq a'_t$, as

$$a_t = c_t + E_t \frac{p_{t+1}}{p_t} a_{t+1} - e_t > c'_t + E_t \frac{p_{t+1}}{p_t} a'_{t+1} - e_t = a'_t.$$

We proved that $c'_t \geq c_t$. Clearly $a'_{t+1} \geq a_{t+1}$ on the set $\{a_{t+1} = \phi_{t=1}\}$. On $\{a_{t+1} > \phi_{t=1}\}$, agent's first order conditions are

$$\frac{u'_t(c_t)}{V'_{t+1}(a_{t+1})} = \frac{p_t}{p_{t+1}} \leq \frac{u'_t(c'_t)}{V'_{t+1}(a'_{t+1})},$$

implying that $a'_{t+1} \geq a_{t+1}$, as required. $\square$

Proof of Proposition 3.4

Proof. Let $(c, a) \in C_t(\phi_t, \phi, p)$. Aggregation of agent's budget constraints gives

$$E_t \sum_{s=t}^{\eta_n - 1} p_s c_s = E_t \sum_{s=t}^{\eta_n - 1} p_s e_s + p_t \phi_t - E_t p_{\eta_n} \phi_{\eta_n} - E_t p_{\eta_n} \left(a_{\eta_n} - \phi_{\eta_n}\right). \tag{A.1}$$

Using the inequality $u'(x)x \leq u(x) - u(0) \leq \bar{u} - \underline{u}$ and letting $\bar{U} := \frac{p_t(\bar{u}-\underline{u})E_t \sum_{s\geq t}\beta_s}{\beta_t u'(c_t)}$,

$$0 < E_t \sum_{s=t}^{\eta_n-1} p_s c_s = \frac{p_t}{\beta_t u'(c_t)} \cdot E_t \sum_{s=t}^{\eta_n-1} \beta_s u'(c_s) c_s \leq \bar{U} < \infty. \tag{A.2}$$

Since $(c,a) \in C_t(\phi_t, \phi, p)$, by the transversality condition (Bidian and Bejan 2012, Lemma 1.1),

$$\begin{aligned} &\lim_{n\to\infty} E_t p_{\eta_n} (a_{\eta_n} - \phi_{\eta_n}) = \lim_{n\to\infty} E_t p_n (a_n - \phi_n) \mathbf{1}_{n<\alpha(t)} \qquad \text{(A.3)} \\ &= \lim_{n\to\infty} \frac{p_t}{u_t'(c_t)} E_t u_n'(c_n)(a_n - \phi_n)\mathbf{1}_{n<\alpha(t)} \leq \frac{p_t}{u_t'(c_t)} E_t u_n'(c_n)(a_n - \phi_n) = 0. \end{aligned}$$

From (A.1)-(A.3),

$$-p_t\phi_t + \lim_{n\to\infty} E_t \sum_{s=t}^{\eta_n-1} p_s c_s = \lim_{n\to\infty} \left(E_t \sum_{s=t}^{\eta_n-1} p_s e_s - E_t p_{\eta_n} \phi_{\eta_n} \right) \leq \bar{U} - p_t\phi_t. \tag{A.4}$$

It follows that

$$E_t \sum_{s=t}^{\alpha(t)-1} p_s e_s := \lim_{n\to\infty} E_t \sum_{s=t}^{\eta_n-1} p_s e_s \leq \bar{U} - p_t\phi_t + \sup_{n\geq t} E_t p_{\eta_n}\phi_{\eta_n} < \infty.$$

Moreover, (A.1) in conjunction with (A.2)-(A.3) show that $\inf_{n\geq t} E_t p_{\eta_n}\phi_{\eta_n} > -\infty$, and therefore (since $\sup_{n\geq t} E_t (p_{\eta_n}\phi_{\eta_n})^+ < \infty$)

$$\sup_{n\geq t} E_t \left(p_{\eta_n}\phi_{\eta_n}\right)^- < \infty. \tag{A.5}$$

At any period $s \in \mathbb{N}$, since $B_s(\phi_s, \phi, p) \neq \emptyset$, the agent can consume at least 0 if his beginning of period s wealth is ϕ_s and he faces the bounds ϕ. Thus $p_s\phi_s + p_s e_s \geq E_s p_{s+1}\phi_{s+1}$. It follows that $\left(p_s\phi_s + E_s \sum_{n=s}^{\alpha(t)-1} p_n e_n\right)_{s=t}^{\alpha(t)}$ is supermartingale, which converges by (A.5) (Kopp 1984, Corollary 2.6.2). Therefore $(p_{\eta_n}\phi_{\eta_n})_n$ converges a.s. Similarly, $\left(p_s\bar{\phi}_s + E_s \sum_{n=s}^{\alpha(t)-1} p_n e_n\right)_{s=t}^{\alpha(t)}$ is a supermartingale and we infer that $(p_{\eta_n}\bar{\phi}_{\eta_n})_n$ converges a.s. Hence $(M_{\eta_n})_n$ converges a.s. $\square$

B Omitted proofs in Section 5

Proof of Proposition 5.1

Proof. An analysis of the function $\bar{f}(x) := f(x,x)$ reveals that it is convex (as sum of convex functions), $\bar{f}(0) = 0$ and $\bar{f}(y^H - y^L) > 0$. Moreover $\bar{f}'(x) = u'(y^H - x) - \beta u'(y^L + x)$, and therefore $\bar{f}'$ is strictly increasing. Notice that $\bar{f}'((y^H - y^L)/2) > 0$, and, by (5.1), $\bar{f}'(0) < 0$. Therefore there exists a unique $x^* \in \left(0, \frac{y^H - y^L}{2}\right)$ such that $\bar{f}'(x^*) = 0$, that is, x^* satisfies (5.10). The function $\bar{f}$ decreases strictly up to x^* and then increases strictly. It follows that there exists a (unique) $\bar{x} \in (x^*, y^H - y^L)$, such that $\bar{f}(\bar{x}) = f(\bar{x}, \bar{x}) = 0$.

Given $0 < x_t < \bar{x}$, since $f(x_t, 0) > 0$, $f(x_t, x_t) = \bar{f}(x_t) < 0$ and $f(x_t, \cdot)$ is strictly decreasing, it follows that the equation $f(x_t, x_{t+1}) = 0$ has a unique solution x_{t+1}, which moreover satisfies $0 < x_{t+1} < x_t$. Therefore the sequence (x_t) satisfying $f(x_t, x_{t+1}) = 0$ for all t is strictly decreasing if $0 < x_0 < \bar{x}$. Moreover, the continuity of f implies that $f(\lim x_t, \lim x_t) = \bar{f}(\lim x_t) = 0$, and thus $\lim x_t = 0$. If $x_t \in \{0, \bar{x}\}$, then the solution of $f(x_t, x_{t+1}) = 0$ is $x_{t+1} = x_t$, thus (x_t) is constant if $x_0 \in \{0, \bar{x}\}$.

Construct the prices (p_t) starting from $p_0 := 1$ and using (5.4). The participation constraints of high-type agents are satisfied by the construction of the sequence (x_t). The continuation utilities of low-type agents at a period t exceed the autarchy levels (autarchy being the outside option) since they receive a positive transfer $x_t > 0$ at t, and starting from $t+1$ they will receive a continuation utility equal to autarchy (since they will be high-type next period). The first order condition of the low-type agents are satisfied since (5.6) holds. Indeed, $x_t + x_{t+1} \leq 2x_t \leq y^H - y^L$.

From the agents' budgets constraints, the asset holdings supporting the desired transfers (x_t), taking as given the initial wealth a_0^i of each agent i are $a_{t+1}^e = a_0^e + L(t)/p_{t+1}$, for all $t \geq 0$, and $a^o = -a^e$. The asset holdings of the high-type agents equal their debt limits, and therefore $\phi_{2t}^e = a_{2t}^e$, $\phi_{2t+1}^o = a_{2t+1}^o$, for all $t \geq 0$. To determine ϕ_{2t+1}^e, let $(c', a') \in C_{2t+1}^e(\phi_{2t+1}^e, \phi^e, p)$. The debt constraints of the even agent are binding at $2t+2$ along the path a', since they bind at $2t+2$ on the path a, and his wealth at $2t+1$ on path a is higher than on path a' ($a_{2t+1}^e \geq \phi_{2t+1}^e = a'_{2t+1}$). Thus $a'_{2t+2} = \phi_{2t+2}^e$ and $V_{2t+1}^{e,d} = V_{2t+1}^e(\phi_{2t+1}^e, \phi^e, p) = u_{2t+1}(c'_{2t+1}) + V_{2t+2}^{e,d}$. As the penalty for default is autarchy, $V_{2t+1}^{e,d} = u_{2t+1}(y^L) + V_{2t+2}^{e,d}$, and therefore $c'_{2t+1} = y^L$.

Since $p_{2t+1}c'_{2t+1} + p_{2t+2}a'_{2t+2} = p_{2t+1}y^L + p_{2t+1}\phi^e_{2t+1}$, we infer that

$$p_{2t+1}\phi^e_{2t+1} = p_{2t+2}a'_{2t+2} = p_{2t+2}\phi^e_{2t+2}. \tag{B.1}$$

Similarly, $p_{2t}\phi^o_{2t} = p_{2t+1}\phi^o_{2t+1}$. Notice that for all $t \geq 0$, $\phi^e_t + \phi^o_t = x_t$, since

$$\begin{aligned} \phi^e_{2t} + \phi^o_{2t} &= a^e_{2t} - \frac{p_{2t+1}}{p_{2t}}a^e_{2t+1} = -\frac{(-1)^{2t}p_{2t}x_{2t}}{p_{2t}} = -x_{2t}, \\ \phi^e_{2t+1} + \phi^o_{2t+1} &= \frac{p_{2t+2}}{p_{2t+1}}a^e_{2t+2} - a^e_{2t+1} = \frac{(-1)^{2t+1}p_{2t+1}x_{2t+1}}{p_{2t+1}} = -x_{2t+1}. \end{aligned}$$

Next we determine the restrictions needed on the initial wealth of the agents such that the debt bounds are nonpositive. This is clearly the case when $x_t = 0$ for all t (the autarchic equilibrium), since asset holdings and debt bounds are zero. For non-autarchic equilibria, that is for nonzero sequences (x_t), by (5.12) and the strict concavity of u,

$$\begin{aligned} u'(y^H - x_t)x_t > u(y^H) - u(y^H - x_t) = \beta\left(u(y^L + x_{t+1}) - u(y^L)\right) > \beta u'(y^L + x_{t+1})x_{t+1} \\ u'(y^H)x_t < u(y^H) - u(y^H - x_t) = \beta\left(u(y^L + x_{t+1}) - u(y^L)\right) < \beta u'(y^L)x_{t+1}. \end{aligned}$$

Therefore by (5.4),

$$\frac{p_t}{p_{t+1}} = \frac{u'(y^H - x_t)}{\beta u'(y^L + x_{t+1})} > \frac{x_{t+1}}{x_t} > \frac{u'(y^H)}{\beta u'(y^L)}. \tag{B.2}$$

It follows that $p_t x_t > p_{t+1}x_{t+1}$, and therefore the sequence $(p_t x_t)$ is strictly decreasing. Thus for $i \in \{e, o\}$, the sequences $(p_t\phi^i_t)$ are nondecreasing, hence $p \cdot \phi^e$ and $p \cdot \phi^o$ are indeed submartingales, in agreement to Theorem 4.4. As a consequence, the necessary and sufficient condition for $\phi^i \leq 0$ is $\lim_{t\to\infty} p_t\phi^i_t \leq 0$. Notice that

$$\lim_{t\to\infty} p_t\phi^o_t = \lim_{t\to\infty} p_{2t+1}a^o_{2t+1} = a^o_0 - L_2, \quad \lim_{t\to\infty} p_t\phi^e_t = \lim_{t\to\infty} p_{2t}a^e_{2t} = -a^o_0 + L_1.$$

The limits $L_1 := \lim_{t\to\infty}\sum_{s=0}^{2t-1}(-1)^s p_s x_s$ and $L_2 := \lim_{t\to\infty}\sum_{s=0}^{2t}(-1)^s p_s x_s$ are well-defined and $L_1 \leq L_2$, since $(p_t x_t)$ is decreasing. Therefore $\lim_{t\to\infty} p_t\phi^o_t \leq 0$ and $\lim_{t\to\infty} p_t\phi^o_t \leq 0$ if and only if $L_1 \leq a^o_0 \leq L_2$.

For the given net savings, prices, trading strategies and debt limits to form an AJ-equilibrium, all that is left is to check the (necessary and) sufficient transversality conditions. They are clearly satisfied, as for $i \in \{e, o\}$,

$$\lim_{t\to\infty} \beta^t u'(c_t^i)(a_t^i - \phi_t^i) \leq \lim_{t\to\infty} \beta^t u'(c_t^i) \left(\sum_i a_t^i - \sum_i \phi_t^i \right) = \lim_{t\to\infty} \beta^t u'(c_t^i) x_t = 0. \quad \text{(B.3)}$$

□

Proof of Proposition 5.3

Proof. For each $\pi > 0$, $h(\cdot, \pi)$ is continuous and strictly decreasing in x. Moreover, for each $\pi \in [1, \beta u'(y^L)/u'(y^H)]$, $h(0, \pi) \leq 0$ and $h(x^*, \pi) \geq 0$, and therefore there exists a unique $x(\pi) \in [0, x^*]$ such that $h(x(\pi), \pi) = 0$. In particular, $x(1) = x^*$ and $x(\beta u'(y^L)/u'(y^H)) = 0$. By the implicit function theorem,

$$x'(\pi) = \frac{1}{\pi^2} \frac{u'(y^H - x(\pi)) \cdot \left(\pi u'(y^L + x(\pi)/\pi) + u''(y^L + x(\pi)/\pi) x(\pi)\right)}{u'(y^L + x(\pi)/\pi) \cdot u''(y^H - x(\pi)) + u'(y^H - x(\pi)) \cdot u''(y^L + x(\pi)/\pi)/\pi}.$$

The denominator in the right hand side of the above equation is negative, and

$$\pi u'(y^L + x(\pi)/\pi) + u''(y^L + x(\pi)/\pi) x(\pi) =$$
$$= \pi u'(y^L + x(\pi)/\pi) \left(1 - \frac{-u''(y^L + x(\pi)/\pi)}{u'(y^L + x(\pi)/\pi)} (y^L + x(\pi)/\pi) \frac{x(\pi)/\pi}{y^L + x(\pi)/\pi} \right).$$

By the assumption on the coefficient of relative risk aversion of u,

$$\frac{-u''(y^L + x(\pi)/\pi)}{u'(y^L + x(\pi)/\pi)} (y^L + x(\pi)/\pi) \frac{x(\pi)/\pi}{y^L + x(\pi)/\pi} \leq \frac{x^* + y^L}{x^*} \frac{x(\pi)/\pi}{y^L + x(\pi)/\pi} \leq 1,$$

with strict inequalities if $\pi \in (1, \beta u'(y^L)/u'(y^H))$.

Therefore $x(\cdot)$ is strictly decreasing. The sequences $(x_t), (\pi_t)$ are determined starting from x_0 and using $\pi_t = x^{-1}(x_t)$ and $x_{t+1} = x_t/\pi_t$. When $x_0 = x^*$ it follows that $\pi_0 = 1$, hence $x_t = x^*$ and $\pi_t = 1$ for all t. When $x_0 = 0$ it follows that $\pi_0 = \beta u'(y^L)/u'(y^H)$, hence $x_t = 0$ and $\pi_t = \beta u'(y^L)/u'(y^H)$ for all t. When $0 < x_0 < x^*$ then $x_t \searrow 0$ and $\pi_t \nearrow \beta u'(y^L)/u'(y^H)$, it follows that $\pi_0 \in (1, \beta u'(y^L)/u'(y^H))$, therefore $x_1 = x_0/\pi_0 < x_0$, $\pi_1 = x^{-1}(x_1) > x^{-1}(x_0) = \pi_0$, and it follows immediately that $x_t \searrow 0$ and $\pi_t \nearrow \beta u'(y^L)/u'(y^H)$.

We verify now that $(p, (c^i), (a^i), (\phi^i), (V^{i,d}))$ is an AJ-equilibrium sustaining the transfers (x_t) and bond prices (π_t). The first order conditions of the unconstrained agents (high-type) are satisfied by construction, since $h(x_t, \pi_t) = 0$. The first order conditions of the borrowing constrained agents (low-type) are satisfied since (5.6) holds, because $x_t + x_{t+1} \leq 2x^* \leq y^H - y^L$. Agents' transversality conditions can be checked as in (B.3), while their budget constraints are satisfied by the construction of the debt limits ϕ^o, ϕ^e (see (5.17)). Agents' participation constraints are satisfied since $p \cdot \phi^o, p \cdot \phi^e$ are constant (martingales), and hence NTT. □

Proof of Proposition 5.4

Proof. Notice that $g(0) = 0$, $g'(0) = (u'(y^H))^2/(u'(y^H)+\beta u'(y^L)) > 0$, and $g(x^*) < 0$, by (5.22), hence there exists $\hat{x} \in (0, x^*)$ such that $g(\hat{x}) = 0$. For *any* $\phi^L < 0$, transfers $\hat{x}$, asset holdings $\hat{a}$, bond prices $\hat{\pi}$, and debt limits $\phi^H = -\hat{a}$ for high-types satisfy market clearing conditions, the first order conditions (5.4) and (5.5) (or equivalently, (5.6)), and the transversality conditions, since

$$\lim_{t\to\infty} \beta^t u'(c_t^i)(a_t^i - \phi_t^i) \leq \lim_{t\to\infty} \beta^t u'(c_t^i)(\sum_i a_t^i - \sum_i \phi_t^i) = \lim_{t\to\infty} \beta^t u'(c_t^i)(-\phi^L - \phi^H) = 0.$$

Moreover, ϕ^H satisfies by construction the NTT condition (for a high-type agent), irrespective of $\phi^L < 0$. All that is left is to establish the existence of a $\phi^L < 0$ that satisfies the NTT condition (for a low-type agent).

Assume that i is the low-type agent at t. If ϕ^i were NTT, $V_t^i(\phi^L, \phi^i, p) = V_t^{i,d}$. By the monotonicity property (3.8), debt limits bind at $t+1$ in the problem $P_t^i(\phi^L, \phi^i, p)$ since they bind in the problem $P_t^i(\hat{a}, \phi^i, p)$. It follows that a necessary and sufficient condition for ϕ^i to be NTT is that ϕ^L satisfies

$$u_t(y^L + \phi^L + \hat{\pi}\hat{a}) + V_{t+1}^i(-\hat{a}, \phi^i, p) = u_t(y^L) + V_{t+1}^i(0, \phi^i, p). \tag{B.4}$$

Let $\zeta(\phi^L) := u_t(y^L + \phi^L + \hat{\pi}\hat{a}) + V_{t+1}^i(-\hat{a}, \phi^i, p) - u_t(y^L) - V_{t+1}^i(0, \phi^i, p)$. Notice that for any $\phi^L < 0$, $V_{t+1}^i(-\hat{a}, \phi^i, p) = \beta^{t+1}(u(y^H - \hat{x}) + \beta u(y^L + \hat{x}))/(1 - \beta^2)$ and it does not depend on ϕ^L. Moreover, $V_{t+1}^i(0, \phi^i, p)$ is nonincreasing in ϕ^L. Therefore ζ is strictly increasing in the domain $\mathbb{R}_-$. Concavity of V_{t+1}^i in the first argument and the envelope theorem imply

$$\zeta(\phi^L) \geq u_t(y^L + \phi^L + \hat{\pi}\hat{a}) - u_t(y^L) - \frac{\partial V^i_{t+1}(-\hat{a}, \phi^i, p)}{\partial a} \cdot \hat{a}$$
$$= u_t(y^L + \phi^L + \hat{\pi}\hat{a}) - u_t(y^L) - u'_{t+1}(y^H - \hat{x}) \cdot \hat{a},$$

where the partial derivative refers to the first argument. Therefore

$$\beta^{-t}\zeta(\phi^L) \geq u(y^L + \phi^L + \hat{\pi}\hat{a}) - u(y^L) - \beta u'(y^H - \hat{x}) \cdot \hat{a}. \tag{B.5}$$

As $y^L + \hat{\pi}\hat{a} < y^L + (1 + \hat{\pi})\hat{a} = y^L + \hat{x}$,

$$u(y^L + \hat{\pi}\hat{a}) - u(y^L) > u'(y^L + \hat{\pi}\hat{a}) \cdot \hat{\pi}\hat{a} > u'(y^L + \hat{x}) \cdot \hat{\pi}\hat{a} = \beta u'(y^H - \hat{x}) \cdot \hat{a}.$$

Therefore by (B.5), $\zeta(0) > 0$. It is immediate to see that $\zeta(-\hat{\pi}\hat{a}) < 0$. As a consequence, there exists a unique $\phi^L < 0$ satisfying (B.4). Moreover, $\phi^L > \hat{\pi}\hat{a}$. In fact, if (5.26) holds, $\zeta(-\hat{\pi}\hat{a}) > 0$ and therefore $\phi^L \leq -a/\hat{\pi}$, or equivalently $\phi^H \geq \hat{\pi}\phi^L$. Thus under (5.26), discounted debt limits are supermartingales. □

C Efficiency of the equilibria of Section 5

In order to discuss the efficiency of the equilibria constructed in Proposition 5.1, we introduce first some definitions. An allocation $c = (c^e, c^o) \in X^I_+$ is *feasible* if $c^e_t + c^o_t = e^e_t + e^o_t (= y^H + y^L)$ for all t, and *individually rational* if $U^i_t(c^i) \geq U^i_t(e^i)$, for all $t \in \mathbb{N}$ and $i \in \{e, o\}$. An allocation $\bar{c}$ *Pareto dominates* allocation c if $U^i(\bar{c}^e) \geq U^e(c^e)$ for $i \in \{e, o\}$, with at least one strict inequality. A feasible and individually rational allocation c is *constrained inefficient* if it is Pareto dominated by another feasible and incentive rational allocation $\bar{c}$ (Alvarez and Jermann 2000). An allocation c is *ex-post inefficient* if it is Pareto dominated by an allocation $\bar{c}$ satisfying $U^i_t(\bar{c}^i) \geq U^i_t(c^i)$ and $\sum_i \bar{c}^i_t \leq \sum_i c^i_t$, for all $t \in \mathbb{N}$ and $i \in \{e, o\}$. Conversely, an allocation is *constrained efficient* (respectively *ex-post efficient*), if it is *not* constrained inefficient (respectively ex-post inefficient). Notice that a feasible and individually rational allocation which is ex-post inefficient is always constrained inefficient.

Each nonstationary equilibrium of Proposition 5.1 associated to a sequence of transfers $x_t \to 0$ has the property that (by (5.4))

$$\frac{p_{t+1}}{p_t} \to \frac{\beta u'(y^L)}{u'(y^H)} > 1,$$

and therefore it satisfies the "modified Cass criterion", which is a sufficient condition for ex-post inefficiency (Bloise and Reichlin 2011, Lemma 2). Therefore all the non-stationary equilibria constructed in Proposition 5.1 are also constrained inefficient. By contrast, the stationary equilibrium is always constrained efficient. Indeed, if (5.9) is violated, the stationary equilibrium associated to transfers $(y^H - y^L)/2$ is actually Pareto optimal. If, instead, (5.9) holds, then in the stationary equilibrium associated to transfers $\bar{x}$, by (5.4) and (B.2),

$$\frac{p_{t+1}}{p_t} = \frac{\beta u'(y^L + \bar{x})}{u'(y^H - \bar{x})} < 1.$$

Therefore the stationary equilibrium violates the "weak modified Cass criterion", which is a necessary condition for constrained inefficiency (Bloise and Reichlin 2011, Lemma 3). Based on this example, it is tempting to equate equilibrium low interest rates with inefficiency of the equilibrium. This would imply that bubbles, which require low interest rates, can only exist in inefficient equilibria. However the equivalence between efficiency of an equilibrium and the presence of high interest rates is not true in general and is a consequence of the stationarity of agents' endowments, as pointed out by Bloise and Reichlin (2011, Appendix B). They construct an efficient stationary equilibrium with low interest rates, in a framework similar to ours, but with nonstationary endowments.

We investigate in what follows the efficiency of the equilibria constructed in Propositions 5.3 and 5.4. The penalties for default now depend on endogenous equilibrium variables such as prices and debt limits, and therefore a definition of constrained inefficiency is not obvious. Following Bloise and Reichlin (2011), we say that an allocation $c = (c^e, c^o) \in X_+^I$ is *individually rational given reservation utilities* $\nu = (\nu^e, \nu^o) \in X^I$ if $U_t^i(c^i) \geq \nu_t^i$, for all $t \in \mathbb{N}$ and $i \in \{e, o\}$. A feasible allocation c is *constrained inefficient given some reservation utilities* $\nu \in X^i$ if it is Pareto dominated by an allocation $\bar{c}$ which is feasible and individually rational given the reservation utilities ν. The nonstationary equilibria of Proposition 5.3 and the stationary equilibrium of Proposition 5.4 are ex-post inefficient, by the modified Cass criterion, as bond prices $p_{t+1}/p_t > 1$ for large enough t (for all t for the equilibrium of Proposition 5.4).[18] The stationary equilibrium of Proposition 5.3, associated to constant transfers x^* and zero interest rates (constant pricing kernel), is *not* constrained inefficient given reservation utilities $(V^{i,d})_{i\in\{e,o\}}$ satisfying (2.5) (interdiction to borrow after default). This follows using an identical argument to the one used by Bloise and Reichlin (2011, Appendix B, Claims 5 and 7).

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