Dynamic Contracting under Imperfect Public Information and Asymmetric Beliefs*

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Abstract

We develop a dynamic principal–agent model to show how imperfect public information and asymmetric beliefs, asymmetric risk attitudes, complementary actions by both parties, and inter-temporal adverse selection arising from the agent's unobservable actions interact to affect optimal dynamic contracts. Our continuous-time formulation of the model, which features both "hidden actions" and "hidden states," leads to a simple characterization of optimal contracts in terms of the solution to a first-order nonlinear ordinary differential equation (ODE). We exploit the properties of the ODE to derive a number of novel implications for the effects of asymmetric beliefs, agency conflicts, two-sided actions, and adverse selection on contractual dynamics. Our results highlight the key role played by asymmetric beliefs in reconciling a number of empirical findings: the "private equity" puzzle; the non-monotonic relation between firm value and incentives, and the ambiguous relation between risk and incentives. We also derive empirically testable implications that alter the intuition gleaned from previous literature that does not incorporate the effects of asymmetric beliefs. The presence of dynamic adverse selection could cause risk-sharing to improve or deteriorate over time depending on the level of agent optimism. Asymmetric beliefs, agency conflicts and adverse selection cause permanent and transient components of risk to have differing effects on compensation and investment schedules.

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1 Introduction

Traditional principal–agent models typically assume symmetric beliefs about parameters that are imperfectly observable to both parties. In some applications, however, the assumption of a common prior may be quite strong. For example, there is empirical and anecdotal evidence to suggest that entrepreneurs are usually more optimistic about the prospects of their startups than venture capitalists. The abilities of firm managers are typically uncertain, and managers' perceptions of their own abilities could differ from those of shareholders. Additionally, in the presence of imperfect public information about a parameter that affects payoffs, the agent has "implicit incentives" to use his unobservable actions to influence the dynamic process of learning about the parameter. In addition to the presence of asymmetric beliefs and risk attitudes, therefore, optimal long-term contracts are also affected by the presence of inter-temporal adverse selection arising from the agent's unobservable actions. In particular, the theoretical study of such settings necessitates the analysis of a dynamic mechanism design problem that features "hidden actions" and "hidden states."

We contribute to the literature by developing a principal-agent model that differs from previous models by integrating five key elements in a unified framework: (i) imperfect public information and asymmetric beliefs; (ii) asymmetric risk attitudes; (iii) two-sided Bayesian learning; (iv) dynamic moral hazard; and (v) dynamic adverse selection. Our continuous-time formulation leads to a simple characterization of optimal contracts in terms of the solution to a first-order nonlinear ordinary differential equation (ODE). We exploit the properties of the ODE to derive a number of novel implications that demonstrate how asymmetric beliefs, agency conflicts, Bayesian learning, and inter-temporal adverse selection interact to affect optimal dynamic contracts. Our results show that asymmetric beliefs play a central role in potentially reconciling a number of empirical findings: the significant levels of investment in venture capital and R&D projects despite their high failure rates or the "private equity" puzzle; the tenuous relation between risk and incentives; and the non-monotonic relation between firm value and incentives. The inter-temporal interplay among heterogeneous beliefs, moral hazard and adverse selection also generates potentially testable implications for the differing effects of permanent and transient components of risk on compensation and investment schedules.

We first develop a discrete–time model in which a risk-averse agent with CARA preferences obtains financing for a project from a risk-neutral principal over a finite time horizon. The project's payoff occurs at the terminal date and the key state variable is the project's observable incremental payoff in each period; the contribution of the principal's investment and the agent's unobservable effort over the period to the project's final payoff. The variance of the incremental payoff is the project's *intrinsic risk*, which is invariant through time. The expected incremental payoff has two components: a fixed component that represents the project's *quality* and a discretionary component that is affected by the principal's investment and the agent's effort.[1] The principal and the agent have imperfect information about the project's quality and have differing priors. They "agree to disagree" about their respective mean assessments; the difference between them is the initial degree of agent optimism. We consider the (empirically relevant) scenario in which the agent is optimistic although our analysis can easily accommodate the general scenario where the agent could be optimistic or pessimistic relative to the principal.

While the principal and the agent could disagree on the mean assessments of project quality, they agree on the variance, which is the project's *transient risk*. The project's transient risk is resolved over time as the parties form posterior assessments of the project's quality in a Bayesian manner. Because the agent's effort is unobservable to the principal, he has implicit incentives to influence the principal's posterior assessments of the project's quality through his "off equilibrium" effort choices. The presence of adverse selection arising from the agent's unobserved effort necessitates the consideration of a general mechanism in which the agent's "type" is his past history of effort choices, and the terms of the contract between the principal and the agent are dynamically contingent on the agent's reports about his type. Although the principal correctly infers the agent's effort choices in equilibrium, the derivation of the equilibrium itself requires the consideration of "off equilibrium" paths on which the agent's posterior beliefs constitute a "hidden" state variable. The dynamic mechanism design problem, therefore, features "hidden actions" and "hidden states."

We derive equilibrium long-term contracts and show that the contractual variables—the principal's investments, the agent's effort, and their payoffs—are determined by the agent's total incentive intensity path. The total incentive intensity at each date is the sum of the agent's *explicit* and *implicit* incentive intensities. The explicit incentive intensity is the sensitivity of the change in the agent's stake to the project's incremental payoff over the period. The agent's implicit incentive intensity reflects his incentives to influence the principal's *future* assessments of the project's quality through his *current* actions. The optimal total incentive

[1] In applications of the framework to "reputation concerns" settings (Holmstrom, 1999), the project's quality could be interpreted as the agent's ability.

intensity at any date maximizes an objective function that has four components: (i) rents due to the agent's optimism; (ii) costs of risk-sharing; (iii) costs arising from adverse selection due to the agent's unobservable effort; and (iv) the payoff arising from the complementarity between effort and investment.

In any period, the agent's total incentive intensity and effort increase with the degree of agent optimism. An increase in the degree of agent optimism increases the extent to which he overvalues the project's future payoffs relative to the principal that can potentially be exploited by increasing the performance-sensitive component of his compensation relative to the performance-invariant component. However, the adverse selection costs at any date increase with the agent's *future* total incentives that has a dampening effect on the agent's *current* total incentives. We show that the positive effects of optimism dominate the countervailing effects of adverse selection.

The effects of the agent's optimism on the principal's investment depend on its magnitude. If the initial degree of agent optimism is below a threshold—the agent is "moderately optimistic"— the principal's investment increases with the degree of agent optimism. If the initial degree of agent optimism exceeds the threshold, however—the agent is "exuberant"— the principal's investment decreases with the degree of agent optimism. When the agent is moderately optimistic, his total incentive intensity is less than one; the agent's optimal incentive intensity in the benchmark "no agency" scenario with symmetric beliefs and universal risk-neutrality. The complementarity of investment and effort makes it optimal for the principal to increase her investment as the degree of agent optimism increases because the agent's effort also increases. When the agent is exuberant, however, his total incentive intensity exceeds one. In this region, investment and effort effectively become "substitutes." As the degree of agent optimism increases, the optimal contract exploits the agent's exuberance by lowering the principal's investments and allowing the project's output to be dominated by the agent's effort. If the agent is sufficiently exuberant, he over-invests effort relative to the benchmark "no agency" scenario that could explain why venture capitalists and entrepreneurs continue to invest in innovative ventures despite their high failure rates.

The non-monotonic variation of the principal's investment with the degree of agent optimism causes the project/firm value to also vary non-monotonically with the degree of agent optimism. Since the agent's total incentives increase with his optimism, the relation between project/firm value and incentives is non-monotonic, which is consistent with empirical evidence

(Morck et al, 1988). Himmelberg et al (1999) argue that cross-sectional empirical analyses of the relation between firm value and incentives are affected by the possibility that they do not appropriately control for unobserved determinants of firm heterogeneity that simultaneously determine firm value and incentives. Our analysis identifies one such unobserved factor—the degree of asymmetry in beliefs—that leads to a non-monotonic relation between firm value and managerial incentives.

We derive the continuous-time model as the (weak) limit of the discrete-time model as the length of each period tends to zero. We show that the agent's total incentive intensity in the continuous-time equilibrium is the solution of a first-order non-linear ordinary differential equation (ODE). We exploit the properties of the ODE to characterize the equilibrium dynamics. The dynamics of equilibrium contracts are influenced by the two-sided Bayesian learning of the principal and the agent. The passage of time lowers the adverse selection costs that depend on the agent's implicit incentives to influence the principal's future assessments of project quality. However, the passage of time also lowers the agent's optimism as he revises his assessments of project quality. The equilibrium dynamics depend on the relative rates of decline of the degree of agent optimism and adverse selection costs.

If the agent is initially moderately optimistic, the effects of the agent's adverse selection costs on incentives dominate those of his optimism. His total incentive intensities and effort, therefore, increase over time. Because investment and effort are complementary in this region, the principal's investments also increase. When the agent is initially exuberant, the effects of optimism dominate. Because the degree of agent optimism declines over time, his total incentive intensities and effort decline. As investment and effort are substitutes in this region, the principal's investments increase to compensate for the decline in the agent's effort. If we were to assume that the agent's effort is *observable* (but non-contractible) so that implicit incentives are absent, the negative effects of the decline of agent optimism over time cause his total incentives to decline over time *regardless* of his degree of optimism. The presence of implicit incentives and inter-temporal adverse selection, therefore, plays a central role in driving the differing dynamics—increasing or decreasing—of total incentives depending on the level of agent optimism.

In dynamic principal-agent models with imperfect information and *symmetric beliefs*, the resolution of uncertainty with the passage of time generally improves risk-sharing. In contrast, our results show that the presence of asymmetric beliefs and adverse selection could cause risk-sharing to improve or deteriorate over time because the passage of time not only resolves uncertainty, but also the agent's optimism. Our results have two additional surprising implications. First, the agent's explicit incentive intensity exceeds his total incentive intensity when he is exuberant. Significant optimism, therefore, causes the agent's implicit incentives to influence the principal's future assessments of the project's quality to be negative. Second, if the duration of the relationship is sufficiently long, the agent's implicit incentives could be so strong as to cause his explicit incentives to be *negative* early in the relationship.

The project's intrinsic and transient risks have complex and differing effects on contractual dynamics. The differing effects arise from the fact that the intrinsic risk is invariant through time and affects the costs of risk sharing. The transient risk, however, affects the intertemporal adverse selection costs and is resolved over time due to the process of Bayesian learning. Furthermore, the intrinsic and transient risks have opposite effects on the speed with which the degree of agent optimism decreases over time. In particular, our results show that, depending on the level of agent optimism, both types of risk could have positive or negative effects on the agent's incentives as well as the principal's investments.

Our results contrast sharply with the predictions of traditional principal-agent models (e.g. Holmstrom and Milgrom, 1987) in which the relation between risk and incentives is negative. Prendergast (1999) highlights the lack of consensus in the empirical literature on whether the relation between risk and incentives is positive or negative. Our study suggests an alternate explanation for the data by showing that the presence of asymmetric beliefs and dynamic adverse selection could lead to an ambiguous relation between risk and incentives.

2 Related Literature

Our study belongs to the body of literature that analyzes dynamic principal-agent models with moral hazard. Holmstrom and Milgrom (1987) present a continuous-time principal-agent framework with CARA preferences and normally distributed payoffs. Schattler and Sung (1993) develop the first-order approach to the analysis of continuous-time principal-agent problems with exponential utility using martingale methods. Spear and Srivastava (1987) develop a recursive approach for the analysis of dynamic principal-agent models with moral hazard that has been subsequently applied and extended to investigate managerial compensation (e.g., Wang, 1997, Spear and Wang, 2005), financial contracting (e.g. Quadrini, 2004, Clementi and Hopenhayn, 2006, DeMarzo and Fishman, 2007a, 2007b), and various issues in macroeconomics (see Part V of Ljungqvist and Sargent, 2004). More recent studies show how to apply recursive techniques to analyze continuous-time principal-agent models (e.g. Cadenillas, Cvitanic and Zapatero, 2007, Biais et al, 2007, Sannikov, 2008, Williams, 2009, Biais et al, 2010). We contribute to this literature by analyzing a model with heterogeneous beliefs, hidden actions and hidden states. In addition to the usual tradeoff between risk-sharing and incentives, the optimal dynamic contracts in our framework reflect the effects of Bayesian learning and dynamic adverse selection.

Gibbons and Murphy (1992) adapt the framework of Holmstrom (1999, formerly 1982) to study how reputation concerns affect incentive contracts for workers and predict that incentive intensities increase with experience. Meyer and Vickers (1997) develop a dynamic model with explicit and implicit incentives and examine the effects of relative performance evaluation. Zwiebel (1995) shows that reputation concerns could lead to conservative behavior by managers. Prendergast and Stole (1996) show that, in a dynamic setting with reputation concerns, agents could invest aggressively in early periods, but ultimately become too conservative. None of these studies focuses on the effects of asymmetric beliefs.

Landier and Thesmar (2009) examine the effects of optimism in a two-period model in which the principal and agent are risk-neutral and the contractual space is exogenously restricted to debt contracts. They show that optimistic entrepreneurs are more likely to choose short-term debt. Adrian and Westerfield (2009) analyze a continuous-time principal-agent model with heterogeneous beliefs. They show that asymmetric beliefs create disagreement risk that alters optimal risk-sharing. For tractability, they assume that the agent "cannot learn in secret" so that his posterior beliefs are observable to the principal and, therefore, do not constitute a hidden state variable. We also complement their study by incorporating dynamic actions by both parties and two-sided Bayesian learning.

The discrete-time model that we develop (before moving to continuous-time) is based on the framework of Giat et al (2010). For tractability, they assume that the agent's actions are observable (but non-contractible) so that his posterior beliefs are observable to the principal. As discussed earlier, the lack of observability of the agent's actions necessitates the analysis of a more complex dynamic mechanism design problem with hidden actions and hidden states. In addition to the effects of asymmetric beliefs and risk-sharing, the optimal contracts in our framework also reflect the effects of implicit incentives and dynamic adverse selection.

3 A Discrete-Time Model

We begin by analyzing a discrete-time model. The continuous-time model that we present later in Section 5 is the limit (in a rigorously defined sense) of the discrete-time model as the time interval between successive dates tends to zero. The presence of imperfect public information, Bayesian learning, and persistent private information for the agent arising from his unobserved actions makes it very difficult to rigorously pursue a *direct* analysis of the continuous–time model in our setting as in studies such as Sannikov (2008). Further, the investigation of the discrete–time model helps to clarify the intuition underlying the properties of the continuous–time equilibrium.

We consider a finite time horizon $[0, T]$ with dates $0, \Delta t, 2\Delta t, ..., T - \Delta t, T$. At date zero, an agent with a project approaches a principal for investments in the project. The project generates value through physical capital investments by the principal and human capital investments (effort) by the agent in each period.

3.1 The Project's Payoff

The project's payoff, which occurs at the terminal date T, is

$$V_T = V_0 + \sum_{t=0}^{T-\Delta t} \Delta V_t.$$

In the above ΔV_t is the contribution to the project's final payoff from the principal's and agent's actions in the interval $[t, t+\Delta t]$, which we hereafter refer to as the project's *incremental payoff* over the period $[t, t + \Delta t]$.[2]

The incremental payoff, ΔV_t, is the sum of a base output—a normal random variable that is unaffected by the actions of the principal and agent—and a discretionary output—a deterministic component that depends on the principal's investment and the agent's effort. It is given by

$$\Delta V_t = \overbrace{\Theta \Delta t + s\left(W_{t+\Delta t} - W_t\right)}^{\text{Base Output}} \quad + \quad \overbrace{\Phi(c_t, \eta_t)\Delta t.}^{\text{Discretionary Output}} \tag{1}$$

The first component, Θ, of the base output represents the project's core output growth rate, which we refer to as the project's *intrinsic quality.* In applications of our framework to

[2]We can extend the model to incorporate intermediate cash flows that are proportional to the project's incremental payoffs without altering our results.

settings with "reputation concerns," Θ could be interpreted as the ability of the agent. The principal and agent have imperfect information about Θ and could also differ in their beliefs about its value. Their respective beliefs are, however, common knowledge, that is, they agree to disagree (see Morris, 1995, Allen and Gale, 1999). The principal's (Pr) and agent's (Ag) initial priors on Θ are normally distributed $N(\mu_0^{Pr}, \sigma_0^2)$ and $N(\mu_0^{Ag}, \sigma_0^2)$, respectively.

We make no assumptions about the *true* project quality distribution because the equilibrium does not depend on it. We can consider the general scenario in which the agent could be optimistic or pessimistic relative to the principal. To simplify the exposition, and keeping in mind the canonical applications of our framework (e.g. venture capital, R&D), we restrict consideration to the empirically relevant scenario in which the agent is optimistic relative to the principal so that μ_0^{Ag} is greater than or equal to μ_0^{Pr}. We define $\Omega_0 = \mu_0^{Ag} - \mu_0^{Pr}$ as the *degree of agent optimism* at date zero.

The second component of the base output, $s\left(W_{t+\Delta t} - W_t\right) = s\Delta W_t$, where $s > 0$ is a constant and W is a standard Brownian motion, represents the "intrinsic" component of the project's risk in period $[t, t + \Delta t]$. It is the component of the project's risk that remains invariant over time, and is independent of Θ. We refer to s^2 as the project's *intrinsic risk.*

The discretionary output in period $[t, t + \Delta t]$ is a direct result of the principal's capital investment rate c_t and the agent's effort η_t, and is given by

$$\Phi(c_t, \eta_t) = Ac_t^{\alpha}\eta_t^{\beta}, \quad \alpha, \beta > 0. \tag{2}$$

The agent's effort choices are *unobservable* to the principal, but are correctly inferred by the principal *in equilibrium.*

The principal and the agent update their prior beliefs of the project's intrinsic quality, Θ, over time based on intermediate observations of the project's incremental payoffs. Because the agent's effort is unobservable to the principal, the principal's posterior assessments of the project's quality depend on her *inferences* of the agent's past effort. Let η_t^{Pr} denote the principal's inference of the agent's effort over the interval $[t, t + \Delta t]$. Let η_t^{Ag} denote the agent's (possibly off-equilibrium) effort over the interval. It follows from well-known formulae (see Oksendal, 2003) that the principal's and agent's posterior assessments of Θ at each date

$t \in \{\Delta t, ..., T\}$ are normally distributed and denoted as $N\left(\mu_t^{Pr}, \sigma_t^2\right)$ and $N\left(\mu_t^{Ag}, \sigma_t^2\right)$, where

$$\sigma_t^2 = \frac{s^2\sigma_0^2}{s^2 + t\sigma_0^2}, \tag{3}$$

$$\mu_t^{Pr} = \frac{s^2\mu_0^{Pr} + \sigma_0^2 \sum_{u=0}^{t-\Delta t}(\Delta V_t - \Phi(c_t, \eta_t^{Pr})\Delta t)}{s^2 + t\sigma_0^2},$$

$$\mu_t^{Ag} = \frac{s^2\mu_0^{Ag} + \sigma_0^2 \sum_{u=0}^{t-\Delta t}(\Delta V_t - \Phi(c_t, \eta_t^{Ag})\Delta t)}{s^2 + t\sigma_0^2} = \frac{s^2\mu_0^{Ag} + \sigma_0^2 \sum_{u=0}^{t-\Delta t}(\Theta\Delta t + s\Delta W_t)}{s^2 + t\sigma_0^2}. \tag{4}$$

The degree of uncertainty, σ_t^2, in the principal's and agent's posterior assessments of the project's intrinsic quality, Θ, at any date t is the project's *transient risk.* It is resolved over time as the principal and agent update their assessments of Θ.

From (4), we note that, because the agent knows his own effort choices, his posterior assessments of the project's quality do not depend on his prior effort choices. However, because the agent's effort choices affect the evolution of the state variable $V(\cdot)$, it follows from (4) that the agent can influence the principal's posterior assessments of project quality through his "off equilibrium" effort choices. Because the principal correctly infers the agent's effort choices in equilibrium, her posterior assessments do not depend on the agent's prior effort choices on the equilibrium path.

Let Ω_t denote the *equilibrium* degree of agent optimism at date t. By (4),

$$\Omega_t = \mu_t^{Ag} - \mu_t^{Pr} = \Omega_0 \frac{s^2}{s^2 + t\sigma_0^2}. \tag{5}$$

Hence, the equilibrium degree of agent optimism declines deterministically over time. Keep in mind that only the agent knows the true degree of agent optimism *off equilibrium.*

3.2 Contracting

The principal offers the agent a long-term contract at date zero. The contract describes the principal's capital investments in each period and their respective payoffs. The contract could be explicitly contingent on the project's contractible incremental payoffs ΔV_t. As in the traditional principal-agent literature, it is convenient to augment the definition of the contract to also include the agent's recommended effort choices. The contract must then be incentive compatible or implementable with respect to the agent's effort.

Because the agent's effort is unobservable to the principal, there is *adverse selection* between the principal and agent at intermediate dates. The agent's unknown "type" at date $t > 0$ is his sequence of prior effort choices, which affects his posterior assessment, μ_t^{Ag}, of the project's intrinsic quality (see 4). The presence of adverse selection necessitates the consideration of the general mechanism in which the agent sends messages about his type to the principal, and the contract could be dynamically contingent on the agent's messages. Because we consider a contracting environment with full commitment, it follows from the revelation principle that we can restrict consideration to direct revelation mechanisms in which the agent announces his type to the principal.

Let $\{\mathcal{F}_t^{Pr}\}$ denote the principal's information filtration (which is common knowledge) at date $t \in \{0, \Delta t, ..., T\}$ that is generated by the history of the project's incremental payoffs, the principal's investments, and the agent's *announced* effort choices $\{\widehat{\eta}_t^{Ag}\}$. Let $\{\mathcal{F}_t^{Ag}\}; t \in \{0, \Delta t, ..., T\}$ denote the agent's information filtration that is generated by the history of the project's incremental payoffs, the principal's investments, and the agent's *actual* effort choices $\{\eta_t^{Ag}\}$. A contract is described by (Q_T, c, η), where Q_T is $\{\mathcal{F}_T^{Pr}\}$-measurable and c, η are $\{\mathcal{F}_t^{Pr}\}$-adapted processes. Q_T is the principal's contractually promised payoff at date T so that $P_T = V_T - Q_T$ is the payoff to the agent. $c_t \Delta t$ is the principal's investment and η_t is the *recommended effort* of the agent over the interval $[t, t + \Delta t]$.

The principal is risk-neutral whereas the agent is risk-averse with CARA preferences. Their respective discount rates are equal and set to zero to simplify the notation. The agent's expected utility at date zero from a contract (Q_T, c, η) is

$$-E_0^{Ag} \left\{ \exp\left(-\lambda \left[P_T - \sum_{t=0}^{T-\Delta t} k \eta_t^{\gamma} \Delta t \right] \right) \right\}. \tag{6}$$

In (6), the parameter $\lambda \geq 0$ characterizes the agent's risk aversion, and $k\eta_t^{\gamma} \Delta t$ is the agent's disutility of effort in the interval $[t, t + \Delta t]$.

Define the principal's *promised payoff* Q_t at date $t \in \{0, \Delta t, ..., T\}$ as the expected future payoff to the principal less her subsequent capital investments under the contract, that is,

$$Q_t = E_t^{Pr} \left\{ Q_T - \sum_{u=t}^{T-\Delta t} c_u \Delta t \right\}. \tag{7}$$

For future reference, we refer to $P_t = V_t - Q_t$ as the *agent's stake* at date t.

For our analysis, it is convenient to define the *agent's continuation utility ratio* at date t, CUR_t^{Ag} (the superscript indicates that this is the continuation utility ratio under the agent's information and beliefs), which is the ratio of his expected future utility (including disutilities of effort) to the expected utility from his current stake. Since the agent has a negative exponential utility function, his continuation utility ratio is

$$CUR_t^{Ag} = E_t^{Ag}\left\{\exp\left[-\lambda\left(P_T - P_t - \sum_{u=t}^{T-\Delta t} k\eta_u^{\gamma}\Delta t\right)\right]\right\}, \tag{8}$$

where the notation E_t^{Ag} denotes the agent's expectation conditioned on the agent's information and beliefs at date t, that is, the σ-field $\mathcal{F}_t^{Ag}$.

A contract (Q, c, η) is incentive compatible for the agent if and only if it is optimal for the agent to exert effort η and truthfully announce his prior effort choices (or "type") to the principal. A contract is feasible if and only if it is incentive compatible and guarantees the principal the promised payoff, Q_0, at date zero.

We consider the sub-class of contracts in which the change in the principal's promised payoff over any period $[t, t+\Delta t]$ is affine in the project's incremental payoff, that is,

$$\Delta Q_t = -a_t\Delta t + (1 - b_t)\Delta V_t, \tag{9}$$

where a_t and b_t are $\mathcal{F}_t^{Pr}$-measurable random variables. It immediately follows from the above that the change in the agent's stake is

$$\Delta P_t = a_t\Delta t + b_t\Delta V_t. \tag{10}$$

From (10), the parameter b_t represents the sensitivity of the change in the agent's stake to the project's performance ΔV_t that we hereafter refer to as the agent's explicit incentive intensity. The parameter a_t determines the agent's performance-invariant compensation; the component of the change in the stake that does not depend on performance during the period. Note that we allow for both a_t and b_t to be random variables.

It follows by adapting the arguments of Mirrlees (1999, originally 1975) that an optimal contract among the *entire set* of incentive feasible contracts in the discrete-time model does not exist. We, therefore, derive the optimal contract in the continuous-time model by first showing that it must be "locally affine" in Proposition 3 (more precisely, the principal's promised payoff under the optimal contract evolves as an Ito process). We then prove that the optimal contract *in the sub-class of affine contracts* described by (9) and (10) converges to the optimal contract (among all incentive feasible contracts) in the continuous-time model described in Section 5 as the time interval Δt between successive dates tends to zero.

Remark 1

In our continuous–time model, we characterize the set of incentive feasible contracts that maximize the agent's expected utility, while guaranteeing the principal an initial promised payoff, Q_0. Such contracts lie on the "utility possibility frontier." The principal's initial promised payoff represents an allocation of bargaining power between the principal and the agent. Traditional dynamic contracting models with moral hazard characterize optimal contracts by maximizing the principal's expected payoff while guaranteeing the agent a promised expected utility (e.g. Spear and Srivastava, 1987). In our setting with imperfect public information and inter-temporal adverse selection, it is much more convenient to adopt the "reflected" perspective in which we maximize the agent's expected utility while guaranteeing the principal a promised expected payoff.

4 The Discrete-Time Equilibrium

We assume the following condition that guarantees the existence of an equilibrium by ensuring that (i) the agent faces decreasing returns to scale from effort; and (ii) his disutility from his effort is sufficiently pronounced relative to its positive contribution to output.

Assumption 1 $(1-\alpha)\gamma/\beta \geq 2$.

4.1 The Discrete-Time Optimal Contract: A Special Case

Before deriving the discrete–time equilibrium of the full model, it is useful to analyze the special case where the agent's effort is the only factor of production. Specifically, we set $A = 1, \alpha = 0, \beta = 1$ in (2). Further, we set $\gamma = 2$ in (6). The intuition underlying some of the features of the optimal contract in this special case carries over to the general setting. We derive the optimal contract by backward induction.

Optimal Contractual Parameters in the Last Period

Consider the last period $[T-\Delta t, T]$. Suppose that the contractual compensation parameters are (a, b) (see (10)). If the agent exerts effort η, his continuation utility ratio (8) is

$$CUR^{Ag}_{T-\Delta t} = E^{Ag}_{T-\Delta t}\left\{\exp\left[-\lambda\left(a\Delta t + b\Delta V_t - k\eta^2\Delta t\right)\right]\right\}.$$

Using the fact that

$$E^{Ag}_{T-\Delta t}\left[\Delta V_t\right] = (\eta + \mu^{Ag}_{T-\Delta t})\Delta t \quad \text{and} \quad Var^{Ag}_{T-\Delta t}\left[\Delta V_t\right] = s^2\Delta t + \sigma^2_{T-\Delta t}\Delta t^2,$$

where $\mu^{Ag}_{T-\Delta t}$ and $\sigma^2_{T-\Delta t}$ are defined in (4) and (3), respectively, the continuation utility ratio is

$$CUR^{Ag}_{T-\Delta t} = \exp\left[-\lambda\left(a\Delta t + b(\eta + \mu^{Ag}_{T-\Delta t})\Delta t - k\eta^2\Delta t - 0.5\lambda b^2\left(s^2\Delta t + \sigma^2_{T-\Delta t}\Delta t^2\right)\right)\right]. \quad (11)$$

Because the agent has a negative exponential utility function, he chooses the effort level to minimize (11). The agent's effort η is therefore implementable if and only if $\eta = \eta(b)$, where

$$\eta(b) = \frac{b}{2k}. \quad (12)$$

The principal's inferences of the agent's prior effort choices are $\{\eta^{Pr}_u; u < T-\Delta t\}$ so that her mean assessment of the project's intrinsic quality at date $t = T - \Delta t$ is $\mu^{Pr}_{T-\Delta t}$ (given by 4). The derivation of the equilibrium necessitates the consideration of "off equilibrium" paths in the game tree at which the agent's past effort choices could differ from the principal's inferences. Further, we need to consider the general dynamic mechanism in which the agent's announcements regarding his past effort choices could differ from his actual effort choices.

Accordingly, suppose that the agent announces that his prior effort choices are $\{\widehat{\eta}^{Ag}_u; u < T - \Delta t\}$, while his actual effort choices are $\{\eta^{Ag}_u; u < T - \Delta t\}$. Hence, the agent's *actual* mean assessment of the project's intrinsic quality at date $t = T - \Delta t$ is $\mu^{Ag}_{T-\Delta t}$, but his mean assessment based on his *announced* prior effort is $\widehat{\mu}^{Ag}_{T-\Delta t}$. $\mu^{Ag}_{T-\Delta t}$ and $\widehat{\mu}^{Ag}_{T-\Delta t}$ are given by (4) with the agent's prior effort choices set to η^{Ag}_u and $\widehat{\eta}^{Ag}_u$, respectively.

The contract must satisfy the promise-keeping constraint for the principal, that is, the principal's promised payoff must be delivered by the contract *under the principal's beliefs.* We conjecture that, under the optimal contract, the principal does not alter her inferences of the agent's prior effort in response to the agent's announcement. We later verify that our conjecture is correct by showing that the principal correctly infers the agent's effort choices in equilibrium and that the principal's response to the agent's announcements supports the equilibrium. By (7) and (9), the principal's promise-keeping constraint is (recall that effort is the only driver of production in this special case, that is, there is no inter-temporal investment by the principal)

$$E^{Pr}_{T-\Delta t}[\Delta Q_t] = 0. \tag{13}$$

The expectation above assumes that the principal's assessment of the project's intrinsic quality at date $t = T-\Delta t$ is $\mu^{Pr}_{T-\Delta t}$. It follows from (13) and (12) that the agent's performance-invariant compensation a, expressed in terms of the agent's explicit incentive intensity b, is

$$a(b) = (1-b)\left[\frac{b}{2k} + \mu^{Pr}_{T-\Delta t}\right]. \tag{14}$$

Substituting (12) and (14) into (11), the agent's continuation utility ratio simplifies to

$$\begin{aligned} CUR^{Ag}_{T-\Delta t} &= \exp\left[-\lambda\Lambda^{Ag}_{T-\Delta t}(b)\Delta t\right] \\ &= \exp\left[-\lambda\left(\Omega^{Ag}_{T-\Delta t}b - 0.5\lambda(s^2+\sigma^2_{T-\Delta t}\Delta t)b^2 + \frac{b}{2k} - \frac{b^2}{4k} + \mu^{Pr}_{T-\Delta t}\right)\Delta t\right]. \end{aligned} \tag{15}$$

In (15), $\Omega^{Ag}_{T-\Delta t} = \mu^{Ag}_{T-\Delta t} - \mu^{Pr}_{T-\Delta t}$.

Based on the agent's announcement regarding his prior effort choices, however, his "announced" or "apparent" continuation utility ratio is

$$\widehat{CUR}^{Ag}_{T-\Delta t} = \exp\left[-\lambda\widehat{\Lambda}^{Ag}_{T-\Delta t}(b)\Delta t\right] = \exp\left[-\lambda\left(\widehat{\Omega}^{Ag}_{T-\Delta t}b - 0.5\lambda(s^2+\sigma^2_{T-\Delta t}\Delta t)b^2 + \frac{b}{2k} - \frac{b^2}{4k} + \mu^{Pr}_{T-\Delta t}\right)\Delta t\right], \tag{16}$$

where $\widehat{\Omega}^{Ag}_{T-\Delta t} = \widehat{\mu}^{Ag}_{T-\Delta t} - \mu^{Pr}_{T-\Delta t}$. Comparing (16) with (15), note that the true degree of agent optimism, $\Omega^{Ag}_{T-\Delta t}$, is replaced with the "announced" degree of agent optimism, $\widehat{\Omega}^{Ag}_{T-\Delta t}$.

As we justify shortly, under the optimal contract, the agent's explicit incentive intensity minimizes (16) so that it solves

$$\widehat{b}^*_{T-\Delta t} = \arg\max_{b\geq 0} \widehat{\Lambda}^{Ag}_{T-\Delta t}(b). \tag{17}$$

By (17) and the preceding analysis, the agent's explicit incentive intensity minimizes his continuation utility ratio (16) based on his *announcement* regarding his type, while guaranteeing the principal her promised payoff based on her beliefs. That is, the agent appropriates the "announced" surplus from the project. It is, therefore, optimal for the agent to truthfully announce his type so that the contractual parameters maximize the *actual* surplus. Consequently, the contract is incentive compatible for the agent and $\widehat{\Lambda}^{Ag}_{T-\Delta t} = \Lambda^{Ag}_{T-\Delta t}$. The agent's explicit incentive intensity therefore solves

$$\begin{aligned} b^*_{T-\Delta t} &= \arg\max_{b\geq 0} \Lambda^{Ag}_{T-\Delta t}(b) \\ &= \arg\max_{b\geq 0} \Omega^{Ag}_{T-\Delta t} b - 0.5\lambda(s^2 + \sigma^2_{T-\Delta t}\Delta t)b^2 + \frac{b}{2k} - \frac{b^2}{4k} \\ &= \frac{1 + 2k\Omega^{Ag}_{T-\Delta t}}{2k\lambda(s^2 + \sigma^2_{T-\Delta t}\Delta t) + 1} \end{aligned} \tag{18}$$

The other contractual parameters are $a^*_{T-\Delta t} = (1 - b^*_{T-\Delta t})\left[\frac{b^*_{T-\Delta t}}{2k} + \mu^{Pr}_{T-\Delta t}\right]$, $\eta^*_{T-\Delta t} = \frac{b^*_{T-\Delta t}}{2k}$. Because the agent's explicit incentive intensity maximizes his expected utility, while guaranteeing the principal her promised payoff, the contract is incentive efficient in the last period.

Optimal Contractual Parameters in the Penultimate Period

We now examine the penultimate period $[T - 2\Delta t, T - \Delta t]$. Set $t = T - 2\Delta t$. Suppose that the contractual compensation parameters in the period $[t, t+\Delta t]$ are (a, b). If the agent exerts effort η, his continuation utility ratio (8) is

$$\begin{aligned} CUR^{Ag}_t &= E^{Ag}_t\left[\exp\left(-\lambda\left(a\Delta t + b\Delta V_t - k\eta^2\Delta t + a^*_{t+\Delta t}\Delta t + b^*_{t+\Delta t}\Delta V_{t+\Delta t} - k\left(\eta^*_{t+\Delta t}\right)^2\Delta t\right)\right)\right] \\ &= E^{Ag}_t\Bigg[\exp\Bigg(-\lambda\Big(a\Delta t + b(\Theta\Delta t + s\Delta W_t + \eta\Delta t) - k\eta^2\Delta t \\ &\qquad + (1 - b^*_{t+\Delta t})\mu^{Pr}_{t+\Delta t}\Delta t + b^*_{t+\Delta t}(\Theta\Delta t + s\Delta W_{t+\Delta t})\Big)\Bigg)\Bigg], \end{aligned} \tag{19}$$

where the second equality follows from (14) and (1). By (4),

$$\mu^{Pr}_{t+\Delta t} = \frac{s^2\mu^{Pr}_t + \sigma^2_t\left(\Delta V_t - \eta^{Pr}_t\Delta t\right)}{s^2 + \Delta t\sigma^2_t} = \frac{s^2\mu^{Pr}_t + \sigma^2_t\left(\Theta\Delta t + s\Delta W_t + (\eta - \eta^{Pr}_t)\Delta t\right)}{s^2 + \Delta t\sigma^2_t} \tag{20}$$

where η^{Pr}_t is the principal's inference of the agent's effort over the penultimate period $[T - 2\Delta t, T - \Delta t]$. Notice that, at this point in the analysis, we must allow for the "off equilibrium" possibility that the principal's inference of the agent's effort differs from his actual effort.

Substituting (20) in (19), using the fact that $b^*_{t+\Delta t}$ is $\mathcal{F}^{Pr}_t$−measurable by (16), (17), (20), and the fact that the agent truthfully announces his type in the final period, the agent's effort

η is implementable if and only if

$$\eta = \eta(B) = \frac{B}{2k}, \text{where} \tag{21}$$

$$B = b + (1 - b^*_{t+\Delta t})\frac{\sigma_t^2}{s^2 + \Delta t \sigma_t^2}\Delta t. \tag{22}$$

We refer to B as the agent's *total incentive intensity* because it is the sum of his explicit incentive intensity b and his implicit incentive intensity $(1 - b^*_{t+\Delta t})\frac{\sigma_t^2}{s^2+\Delta t\sigma_t^2}\Delta t$. By (21), the agent's effort depends on the total incentive intensity B. Note further that, given the total incentive intensity B, the principal's inference of the agent's effort in the penultimate period, $\eta_t^{Pr} = \eta(B)$.

Let μ_t^{Ag} and $\widehat{\mu}_t^{Ag}$ denote the agent's mean assessments of project quality based on his "actual" and "announced" prior effort choices, respectively. As in our analysis of the final period, we conjecture that the principal does not alter her inferences of the agent's past effort in response to the agent's announcement. Accordingly, the contract must satisfy the promise keeping constraint (13) for the principal, where the principal's current mean assessment of the project's quality is μ_t^{Pr}. Therefore,

$$a(b, c) = (1 - b)\left[\frac{B}{2k} + \mu_t^{Pr}\right]. \tag{23}$$

Substituting (21) and (23) in (19), the agent's actual continuation utility ratio is

$$CUR_t^{Ag}(b, c) = E_t^{Ag}\left(\exp(-\lambda Z)\right), \text{ where}$$

$$Z = \begin{pmatrix} \left[(1-b)\left[\frac{B}{2k} + \mu_t^{Pr}\right] + b(\Theta + \frac{B}{2k}) - \frac{B^2}{4k}\right]\Delta t + bs\Delta W_t \\ +(1-b^*_{t+\Delta t})\mu^{Pr}_{t+\Delta t}\Delta t + b^*_{t+\Delta t}\left[\Theta\Delta t + s\Delta W_{t+\Delta t}\right] \end{pmatrix}. \tag{24}$$

We note that by (20)

$$\begin{aligned} E_t^{Ag}[Z] &= \left[\Omega_t^{Ag} b + \frac{B}{2k} - \frac{B^2}{4k} + \mu_t^{Pr} + \Omega_t^{Ag} b^*_{t+\Delta t}\right]\Delta t. && (25)\\ Var_t^{Ag}[Z] &= \sigma_t^2\left[B_t\Delta t + b^*_{t+\Delta t}\Delta t\right]^2 + s^2\left[B_t^2\Delta t + (b^*_{t+\Delta t})^2\Delta t\right]. && (26) \end{aligned}$$

The agent's continuation utility ratio based on his "announced" effort choices—his "announced" continuation utility ratio—is, however, given by

$$\widehat{CUR}_t^{Ag}(b,c) = \widehat{E}_t^{Ag}\left(\exp(-\lambda Z)\right), \tag{27}$$

where $\widehat{E}_t^{Ag}[\cdot]$ denote the conditional expectation under the agent's "announced" beliefs. $\widehat{E}_t^{Ag}[Z]$ and $\widehat{Var}_t^{Ag}[Z]$ are the mean and variance of Z under the agent's "announced" beliefs, and are given by (25) and (26), respectively, with Ω_t^{Ag} replaced by $\widehat{\Omega}_t^{Ag} = \widehat{\mu}_t^{Ag} - \mu_t^{Pr}$.

As in our analysis of the last period, the agent's explicit incentive intensity minimizes the agent's "announced" continuation utility ratio and is given by

$$\begin{aligned} b_t^* &= \arg\max_b \widehat{\Omega}_t^{Ag} b - 0.5\lambda\Big(\sigma_t^2\left[B + b_{t+\Delta t}^*\right]^2\Delta t + s^2\left[B^2 + (b_{t+\Delta t}^*)^2\right]\Big) + \frac{B}{2k} - \frac{B^2}{4k}. \\ &= \arg\max_b \widehat{\Omega}_t^{Ag} B - 0.5\lambda\Big(\sigma_t^2\left[B + b_{t+\Delta t}^*\right]^2\Delta t + s^2 B^2\Big) + \frac{B}{2k} - \frac{B^2}{4k}. \end{aligned} \tag{28}$$

The second equality above follows from (22) and the fact that $b_{t+\Delta t}^*$ does not depend on b. It follows again from (22) that the optimization problem (28) can be replaced by one that maximizes the agent's total incentive intensity

$$B_t^* = \arg\max_B \widehat{\Omega}_t^{Ag} B - 0.5\lambda\Big(\sigma_t^2\left[B + b_{t+\Delta t}^*\right]^2\Delta t + s^2 B^2\Big) + \frac{B}{2k} - \frac{B^2}{4k}. \tag{29}$$

The optimal explicit incentive intensity b_t^* is obtained from B_t^* using (22).

As in the analysis of the last period, it is incentive compatible for the agent to truthfully announce his type so that his contract maximizes the actual surplus. Hence, $\widehat{\Omega}_t^{Ag} = \Omega_t^{Ag}$. It follows from (29) that

$$B_t^* = \arg\max_B \Omega_t^{Ag} B - 0.5\lambda\Big(\sigma_t^2\left[B + b_{t+\Delta t}^*\right]^2\Delta t + s^2 B^2\Big) + \frac{B}{2k} - \frac{B^2}{4k} = \frac{1 + 2k\Omega_t^{Ag} - \lambda\sigma_t^2 b_{t+\Delta t}^*\Delta t}{2k\lambda(s^2 + \sigma_t^2\Delta t) + 1} \tag{30}$$

In equilibrium, the principal correctly infers the agent's effort in each period. Extending the above analysis by backward induction (see the proof of Theorem 1 in the Appendix), we obtain the following expression for the agent's *equilibrium* total incentive intensity:

$$B_t^* = \frac{1 + \overbrace{2k\Omega_t}^{\text{Effects of Optimism}} - \overbrace{\lambda\sigma_t^2 \sum_{u=t+\Delta t}^{T-\Delta t} B_u^*\Delta t}^{\text{Effects of Implicit Incentives}}}{2k\lambda(s^2 + \sigma_t^2\Delta t) + 1} \tag{31}$$

Because the principal correctly infers the agent's effort choices in equilibrium, the equilibrium degree of agent optimism, Ω_t, appears in the numerator on the right hand side above.

From (31), the agent's total incentive intensity at any date increases with the degree of agent optimism. Optimism causes the agent to overvalue the performance-sensitive component of his compensation relative to the performance-invariant component. The optimal contract exploits this by increasing the sensitivity of the agent's compensation to performance. However, the agent's total incentive intensity at each date is also affected by his *future* total incentives through the second term in the numerator of (31). This term represents the effects of the agent's implicit incentives on his total incentive intensity. The agent's implicit incentives negatively affect his total incentives and the negative effects increase with the agent's future total incentives. As we see shortly, these basic "forces"—the positive effects of optimism and the negative effects of implicit incentives—carry over to the general setting with investment and effort.

4.2 The Discrete-Time Optimal Contract: The General Case

The following theorem describes the discrete-time equilibrium contract in the general model.

Theorem 1 (Discrete-Time Equilibrium)

Let $(a_t^*, b_t^*, c_t^*, \eta_t^*)$ denote the equilibrium contractual parameters in the interval $[t, t+\Delta t]$, where $t \in [0, T-\Delta t]$. Let Ω_t denote the equilibrium degree of agent optimism at date t. Define the sequence B_t^* recursively as follows starting from $t = T - \Delta t$,

$$B_t^* = \arg\max_B \Omega_t B - 0.5\lambda s^2 B^2 - 0.5\lambda\sigma_t^2 \left[B + \sum_{u=t+\Delta t}^{T-\Delta t} B_u^*\right]^2 \Delta t + \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma} c(B), \quad \text{where} \tag{32}$$

$$c(B) = 1_{B<\frac{\gamma}{\beta}} \left[\frac{\alpha\gamma}{\gamma-\beta} A^{\frac{\gamma}{\gamma-\beta}} \left[\frac{1}{k}\right]^{\frac{\beta}{\gamma-\beta}} \left(\frac{\beta B}{\gamma}\right)^{\frac{\beta}{\gamma-\beta}} \left(1 - \frac{\beta B}{\gamma}\right)\right]^{\frac{\gamma-\beta}{(1-\alpha)\gamma-\beta}} \tag{33}$$

The optimal explicit incentive intensity is

$$b_t^* = B_t^* - \sum_{v=t+\Delta t}^{T-\Delta t} (1 - b_v^*) \frac{\sigma_0^2}{\sigma_0^2 v + s^2} \Delta t = B_t^* - \frac{\sigma_0^2}{\sigma_0^2(t+\Delta t) + s^2} \sum_{v=t+\Delta t}^{T-\Delta t} (1 - B_v^*) \Delta t. \tag{34}$$

The principal's optimal investment is

$$c_t^* = c(B_t^*). \tag{35}$$

The agent's optimal effort is

$$\eta_t^* = \left(\frac{A\beta(c_t^*)^\alpha B_t^*}{\gamma k} \right)^{\frac{1}{\gamma-\beta}}. \tag{36}$$

The agent's performance-invariant compensation is

$$a_t^* = (1 - b_t^*) \left[A(c_t^*)^\alpha (\eta_t^*)^\beta + \mu_t^{Pr} \right] - c_t^*. \tag{37}$$

Proof. The proofs of all results are provided in the Appendix.

4.3 The Agent's Total Incentive Intensity

By Theorem 1, the contractual parameters at any date are determined by the agent's total incentive intensity, B_t^* that is given by

$$B_t^* = \overbrace{b_t^*}^{\text{explicit incentive intensity}} + \overbrace{\sum_{v=t+\Delta t}^{T-\Delta t} (1 - b_v^*) \frac{\sigma_0^2}{\sigma_0^2 v + s^2} \Delta t}^{\text{implicit incentive intensity}}. \tag{38}$$

Extending the discussion in Section 4.1, the total incentive intensity is the sum of the agent's "explicit" incentive intensity, b_t^* and the "implicit" incentive intensity. From (32), the agent's optimal total incentive intensity maximizes the objective function $F_t(B)$ that is

$$F_t(B) = \overbrace{\Omega_t B}^{\text{Rents from Agent Optimism}} - \overbrace{0.5\lambda s^2 B^2}^{\text{Costs of Risk-Sharing}} \overbrace{-0.5\lambda\sigma_t^2 \left[B + \sum_{u=t+\Delta t}^{T-\Delta t} B_u^* \right]^2 \Delta t}^{\text{Adverse Selection Costs}} + \overbrace{\frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma} c(B)}^{\text{Return on Investment and Effort}}. \tag{39}$$

The objective function has four key components.

- Rents from Agent Optimism: The term $\Omega_t B$ represents the positive effects of agent optimism on incentives.
- Costs of Risk-Sharing: The term $0.5\lambda s^2 B^2$ represents the usual costs of risk-sharing between the risk-neutral principal and the risk-averse agent that negatively affect total incentives.
- Adverse Selection Costs: The term $0.5\lambda\sigma_t^2 \left[B + \sum_{u=t+\Delta t}^{T-\Delta t} B_u^* \right]^2 \Delta t$ arises from the presence of inter-temporal adverse selection between the principal and the agent because the agent knows his own past effort choices, while the principal doesn't. Consistent with

the intuition gleaned from the analysis of the special case in Section 4.1, the adverse selection costs at any date decline with the agent's *future* total incentives.

- Return on Investment and Effort: The term $\frac{\gamma-\beta-\alpha\gamma}{\alpha\gamma}c(B)$ arises from the complementarity of investment and effort.

The following proposition, which is similar to Proposition 1.1 in Giat et al (2010), establishes some properties of the function $c(.)$ that are important for our subsequent analysis. The proof and the intuition underlying the proposition are as in Giat et al (2010).

Proposition 1
The function $c(B)$ achieves its maximum at $B = 1$. It is strictly concave on $[0, B_M]$ and strictly convex on $[B_M, \frac{\gamma}{\beta}]$, where $B_M \in (1, \frac{\gamma}{\beta}(1-\alpha)\ )$ is the unique minimum of the function $c'(\cdot)$.

We make the following standing assumption, which ensures that the optimal investment function is strictly concave for realized equilibrium values of the total incentive intensity.

Assumption 2 $\Omega_0/\lambda s^2 \leq B_M$*, where* B_M *is defined in Proposition 1.*

The following proposition establishes properties of the optimal total incentive intensity, B_t^*, that we use later.

Proposition 2
(i) The optimal total incentive intensity B_t^* is strictly positive at each date t. (ii) $B_t^* \leq \max\{\frac{\Omega_0}{\lambda s^2},\ 1\}$.

By Proposition 1, $c'(0) = \infty$ so that $F_t'(0) = \infty$. It follows from (39) that the total incentive intensity must be strictly positive at each date. As shown by the second part of the proposition, the total incentive intensity at each date is bounded above with the upper bound determined by the initial degree of agent optimism.

5 The Continuous-Time Model

The continuous-time model is obtained as the limit of the discrete-time model as the time interval between successive dates, Δt, tends to zero. More precisely, the project's terminal payoff, V_T is

$$V_T = V_0 + \int_0^T dV_t, \tag{40}$$

where the incremental payoff V_t evolves as follows:

$$dV_t = \Theta dt + sdW_t \quad + \ \Phi(c_t, \eta_t)dt. \tag{41}$$

The principal's and agent's posterior beliefs about the project's quality, Θ, are similarly given by (3) and (4) where the summations in (4) are replaced by stochastic integrals.

The agent's utility function from a contract (Q_T, c, η) is given by

$$-E_0^{Ag}\left\{\exp\left(-\lambda\left[P_T - \int_0^T k\eta_t^\gamma dt\right]\right)\right\},$$

where $P_T = V_T - Q_T$.

5.1 Structure of Long-Term Contract

The following proposition characterizes the evolution of the principal's promised payoff process under the optimal contract.

Proposition 3

The principal's promised payoff under the optimal contract evolves as follows:

$$dQ_t = -a_t dt + (1 - b_t)dV_t, \tag{42}$$

where the contractual parameters $a_t, b_t \in R$ are $\{\mathcal{F}_t^{Pr}\}$-progressively measurable. It follows directly from (42) that the principal's terminal payoff Q_T can be expressed as

$$Q_T = Q_0 + \int_0^T [-a_t dt + (1 - b_t)dV_t].$$

The agent's stake, $P_t = V_t - Q_t$, at date t evolves as follows:

$$dP_t = a_t dt + b_t dV_t. \tag{43}$$

Analogous to the discrete-time model, we refer to the parameter b_t as the agent's explicit incentive intensity and the parameter a_t as the agent's performance-invariant compensation. In light of Proposition 3 a contract is completely specified by a_t, b_t, and the principal's investment rate, c_t, at each time t. It is worth mentioning here that, by the martingale representation theorem for Brownian motion that we use to prove Proposition 3, the principal's promised payoff under *any* incentive feasible contract must evolve as (42), that is, it must be an Ito process. Therefore, the proposition actually holds for any incentive feasible contract.

5.2 The Continuous-Time Equilibrium

The continuous-time equilibrium is the limit of the equilibrium in the discrete time model described in Theorem 1 as $\Delta t \longrightarrow 0$. Before presenting the main theorem describing the equilibrium, it is useful to describe its derivation using heuristic arguments. From the first order condition in (39) and taking the limit as $\Delta t \longrightarrow 0$, the total incentive intensity in the continuous time model solves

$$F_t^{'}(B_t^*) = \Omega_t - \lambda s^2 B_t^* - \lambda\sigma_t^2 \int_t^T B_u^* du + \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma} c^{'}(B_t^*) = 0 \tag{44}$$

From the above, we obtain

$$\begin{aligned} 0 &= \frac{\Omega_t}{\lambda\sigma_t^2} - \frac{s^2}{\sigma_t^2} B_t^* - \int_t^T B_u^* du + \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma\lambda\sigma_t^2} c^{'}(B_t^*) \\ &= \frac{\Omega_0}{\lambda\sigma_0^2} - \frac{s^2 + t\sigma_0^2}{\sigma_0^2} B_t^* - \int_t^T B_u^* du + \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma\lambda s^2} \frac{s^2 + t\sigma_0^2}{\sigma_0^2} c^{'}(B_t^*), \end{aligned}$$

where the second equality above follows from (3) and (5). Differentiating the second expression above with respect to time and rearranging terms, the total incentive intensity must solve the following nonlinear ODE:

$$\frac{dB_t^*}{dt} = \frac{\sigma_0^2}{s^2 + t\sigma_0^2} \frac{c'(B_t^*)}{\left(\frac{\alpha\gamma\lambda s^2}{\gamma - \beta - \alpha\gamma} - c''(B_t^*)\right)}, \tag{45}$$

The following theorem formally describes the equilibrium and its proof makes the above heuristic derivation rigorous.

Theorem 2 (Continuous-Time Equilibrium)
Let $(a_t^*, b_t^*, c_t^*, \eta_t^*)$ denote the equilibrium contractual parameters at date t. Define the total incentive intensity $B^*(t)$ as the solution of the nonlinear ODE (45) with boundary condition

$$\Omega_T - \lambda s^2 B_T^* + \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma} c'(B_T^*) = 0. \tag{46}$$

where $c(B)$ is defined in (33) at date T.

- *There exists a unique solution to the ODE (45) with boundary condition (46).*
- *The agent's explicit incentive intensity at date t is:*

$$b_t^* = B_t^* - \frac{\sigma_0^2}{s^2 + t\sigma_0^2} \int_t^T (1 - B_u^*) du. \tag{47}$$

- *The principal's equilibrium investment rate at date t is $c_t^* = c(B_t^*)$.*
- *The agent's equilibrium effort at date t is given by (36).*
- *The agent's performance-invariant compensation rate at date t is given by (37).*

As we show in Appendix, the total incentive intensity path in the continuous-time model is the (point-wise) limit of the total incentive intensity paths in the discrete time model as the length of a period tends to zero. Consequently, Proposition 2 is also true for the total incentive intensities in the continuous-time model.

6 Contractual Dynamics

Although the ODE (45) cannot be solved analytically, its properties can be exploited to qualitatively characterize the contractual dynamics.

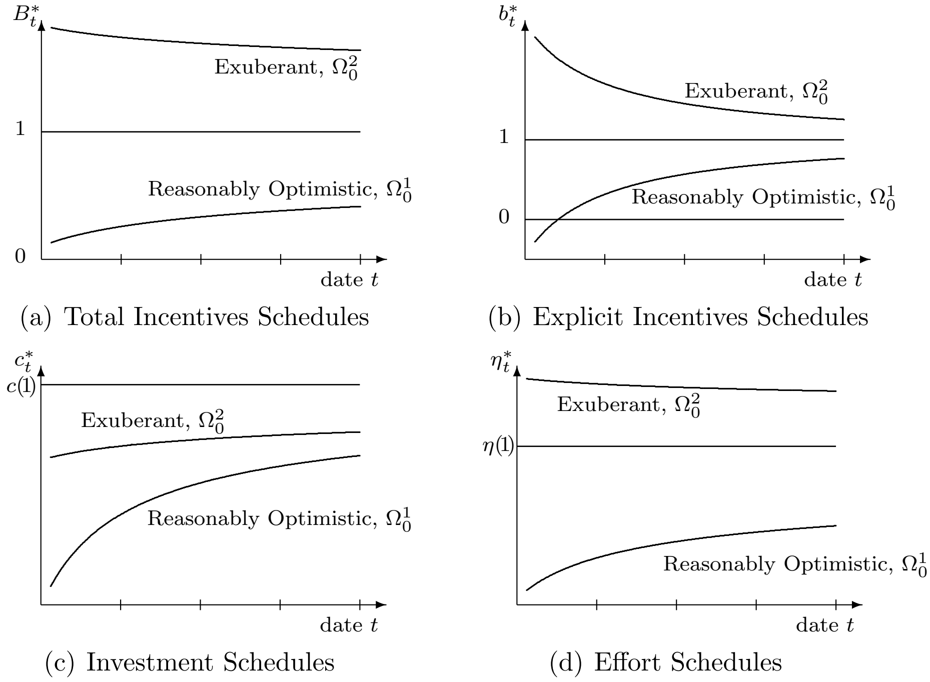
6.1 The Dynamics of Incentives, Effort and Investment

The following theorem describes how the agent's total incentive intensity, explicit incentive intensity, effort, and the principal's investments evolve over time. As the results show, the contractual dynamics crucially depend on the initial degree of agent optimism, specifically, the sign of the quantity $\Omega_0 - \lambda\,(s^2 + T\sigma_0^2)$. We define the agent as "reasonably optimistic" if $\Omega_0 < \lambda\,(s^2 + T\sigma_0^2)$ and "exuberant" if $\Omega_0 > \lambda\,(s^2 + T\sigma_0^2)$.

Theorem 3 (Contractual Dynamics)

- *Suppose that $\Omega_0 < \lambda\,(s^2 + T\sigma_0^2)$. (i) The agent's total incentive intensity $B_t^* < 1$ for all t and B_t^* increases with t. (ii) The agent's explicit incentive intensity $b_t^* < B_t^* < 1$ for all t and b_t^* increases with t. (iii) The investments c_t^* increase over time. (iv) The agent's effort η_t^* increases over time.*

Figure 1: Contractual Dynamics. Here, Ω_0^n, $n = 1, 2$ are initial levels of agent optimism, $\Omega_0^1 < \lambda(T\sigma_0^2 + s^2) < \Omega_0^2$.



- *Suppose that $\Omega_0 = \lambda\left(s^2 + T\sigma_0^2\right)$. (i) The agent's total incentive intensity $B_t^* = 1$ for all t. (ii) The agent's explicit incentive intensity $b_t = 1$ for all t. (iii) The investments c_t^* and effort η_t^* are constant over time.*

- *Suppose that $\Omega_0 > \lambda\left(s^2 + T\sigma_0^2\right)$. (i) The agent's total incentive intensity $B_t^* > 1$ for all t and B_t^* decreases with t. (ii) The agent's explicit incentive intensity $b_t^* > B_t^* > 1$ for all t and b_t^* decreases with t. (iii) The investments c_t^* increase over time. (iv) The agent's effort η_t^* decreases over time.*

Figure 1 illustrates the results of the theorem by showing the variations of the total incentive intensities, explicit incentive intensities, investments and effort choices over time. To understand the intuition for the results, recall the discussion after Theorem 1. The equilibrium depends on the interplay among four forces: agent optimism, risk-sharing costs, inter-temporal adverse selection costs, and the return on investment and effort.

From (39), we note that the effects of agent optimism and the adverse selection costs are explicitly time-dependent, while the costs of risk-sharing and the return on investment are not. By (3) and (5), the transient risk, σ_t^2 and the degree of agent optimism, Ω_t, both decline over time. Hence, by (39), the rents due to the agent's optimism and the adverse selection costs both decline over time. The equilibrium dynamics are essentially determined by the relative rates of decline of these two components of the objective function $F_t(B)$.

When the agent's degree of optimism is below a threshold (that is, he is "reasonably optimistic"), the effects of agent optimism are outweighed by risk-sharing and adverse selection costs so that the agent's total incentive intensity is initially low. As mentioned above, the degree of agent optimism and adverse selection costs both decline over time. When the agent's optimism is below a threshold however, the positive effects of the decline in adverse selection costs on incentives dominate the negative effects of the decline in agent optimism so that the agent's total incentive intensity and effort increase. In the region where $B_t^* < 1$, investment and effort are complementary (recall Proposition 1 and its intuition) so that the principal's investments also increase over time.

When the agent's degree of optimism is above a threshold (that is, he is "exuberant"), the positive effects of agent optimism dominate the negative effects of risk-sharing and adverse selection costs. The agent's total incentive intensity is, therefore, initially high and exceeds one. In this region, the negative effects of the decline in agent optimism on incentives as time passes dominate the positive effects of the decline in adverse selection costs. Hence, the agent's total incentive intensity and effort decline over time. By Proposition 1, the optimal investment function is decreasing for $B > 1$, that is, investment and effort are effectively "substitutes" rather than "complements" in this region. The principal's investments, therefore, increase over time to compensate for the decline in the agent's effort.

A surprising implication of Theorem 3 is that the agent's explicit incentive intensity b_t^* exceeds his total incentive intensity B_t^* when he is exuberant. It follows from (38) that the presence of significant optimism could cause the agent's implicit incentives to influence the principal's posterior assessments about project quality to be negative.

The Effects of Implicit Incentives: By the above discussion, the presence of adverse selection arising from the agent's implicit incentives to influence the principal's posterior assessments of project quality plays a central role in driving the contractual dynamics. To further understand the effects of implicit incentives, we examine the scenario in which they are absent. As discussed earlier, the agent has no implicit incentives when his effort is observable (but non-contractible). In this scenario, the agent's explicit incentive intensity b_t^* at each date t solves

$$\Omega_t b_t^* - \lambda s^2 b_t^* + \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma} c'(b_t^*) = 0 \tag{48}$$

Comparing (48) with (44), we see that, because the agent has no implicit incentives, his explicit incentive intensity equals his total incentive intensity. Further, the explicit incentive intensity maximizes an objective function in which adverse selection costs are absent. The following proposition describes the contractual dynamics in this scenario. We omit the proof for brevity because it follows using arguments similar to those used to prove Theorem 3.

Proposition 4 (Contractual Dynamics When Implicit Incentives are Absent)
(i) The agent's explicit incentive sensitivity b_t^* decreases monotonically with t. Define $t^* := (\frac{\Omega_0}{p} - 1)\frac{s^2}{\sigma_0^2} = \frac{\Omega_0}{\lambda\sigma_0^2} - \frac{s^2}{\sigma_0^2}$. The agent's explicit incentive intensity b_t^* exceeds 1 if $t < t^*$, equals 1 at $t = t^*$, and less than 1 if $t > t^*$. (ii) The investments c_t^* increase until time t^* and then decrease monotonically. (iii) For $t \geq t^*$, the agent's effort η_t^* decreases monotonically. For $t < t^*$, the effort could vary non-monotonically.

Comparing Proposition 4 with Theorem 3, we see that the absence of implicit incentives leads to very different contractual dynamics. In contrast with Theorem 3, incentives always decline over time regardless of the agent's optimism. In the absence of implicit incentives and adverse selection costs, the dynamics of incentives are entirely driven by the fact that the agent's optimism declines over time due to Bayesian learning. The investment and effort paths in the absence of implicit incentives are also very different. In particular, investments and effort vary non-monotonically in general when implicit incentives are absent.

6.2 Sensitivity of Equilibrium Dynamics

We now explore the effects of project characteristics on the contractual dynamics.

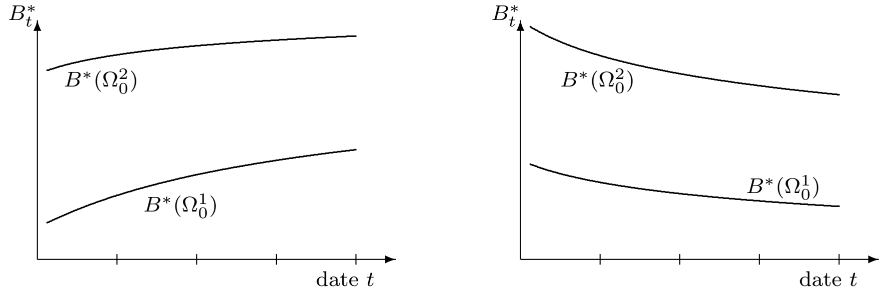
The Effects of Agent Optimism

The following theorem describes the effects of agent optimism on the equilibrium paths of the total incentive intensity, investment and effort.

Theorem 4 (Effects of Agent Optimism)

- *The time paths of total incentive intensities B_t^* and effort η_t^* increase pointwise with the initial degree of agent optimism Ω_0.*
- *If the total incentive intensity path is less (greater) than one (see Theorem 3) then the paths of investments increase (decrease) pointwise with Ω_0.*

Figure 2: Variations of Total Incentive Intensities with Agent Optimism. Here, Ω_0^n, $n = 1, 2, 3, 4$ are initial levels of agent optimism, $\Omega_0^1 < \Omega_0^2 < \lambda(T\sigma_0^2 + s^2) < \Omega_0^3 < \Omega_0^4$.



(a) Reasonably Optimistic Agent (b) Exuberant Agent

By the discussion following Theorem 1, an increase in the degree of agent optimism increases the positive effects of agent optimism on the total incentive intensity. The total incentive intensity at any date is, however, also affected by the three other terms in the objective function (39). In particular, the negative effects of adverse selection costs increase with the *future* total incentive intensities that has a *negative* effect on the current total incentive intensity. As the theorem shows, the *direct* positive effects of agent optimism on total incentives dominates the *indirect* negative effects of adverse selection costs so that the total incentive intensity at each date increases with the initial degree of agent optimism.

Theorem 4 predicts that optimism positively affects the agent's *total incentives* B_t^*. However, the effects of optimism on the agent's *explicit incentives* b_t^* are ambiguous for general parameter values because of the presence of nonzero implicit incentives for the agent. In contrast, in the framework of Giat et al (2009a), in which adverse selection and implicit incentives are absent, agent optimism always positive affects his explicit incentives. The absence of adverse selection in their framework implies that the beneficial effects of optimism are unaffected by the potentially countervailing effects of adverse selection that play a key role in our model.

The effects of agent optimism on investment and effort depend on whether the total incentive intensity path is less than or greater than one because the optimal investment function is non-monotonic as described by Proposition 1. When the total incentive intensities are less than one, investment and effort are complements. Hence, the increase of total incentive intensities with the degree of agent optimism causes the investment and effort paths to also increase. When the total incentive intensities are greater than one, investment and effort are effectively substitutes. Therefore, an increase in the degree of agent optimism increases total incentives and effort, but causes the principal to lower her investments. Put differently, agent optimism is significant in this region so that it is optimal for the principal to allow output to be relatively dominated by the agent's effort.

Our results imply that optimism could significantly mitigate the detrimental effects of risk-sharing and adverse selection costs. In fact, if the degree of agent optimism is sufficiently high, it could even cause him to over-invest effort relative to the benchmark scenario with symmetric beliefs and universal risk-neutrality. In contrast, in traditional principal-agent models with symmetric beliefs, the negative effects of the costs of risk-sharing lead to under-investment of effort relative to the scenario in which the principal and agent are risk-neutral. The significant impact of optimism on investment and effort could reconcile evidence that venture capitalists and entrepreneurs invest in highly innovative ventures even though the chances of failure are extremely high. To explain observed venture capital (VC) investment, a number of previous studies introduce non-pecuniary private benefits for entrepreneurs. The empirical findings in Moskowitz and Vissing-Jorgensen (2002), however, suggest a "private equity" puzzle in that private benefits have to be very high relative to typical entrepreneurial incomes to explain observed levels of VC investment. Our results suggest that entrepreneurial optimism, and its rational exploitation by venture capitalists, could reconcile the puzzle.

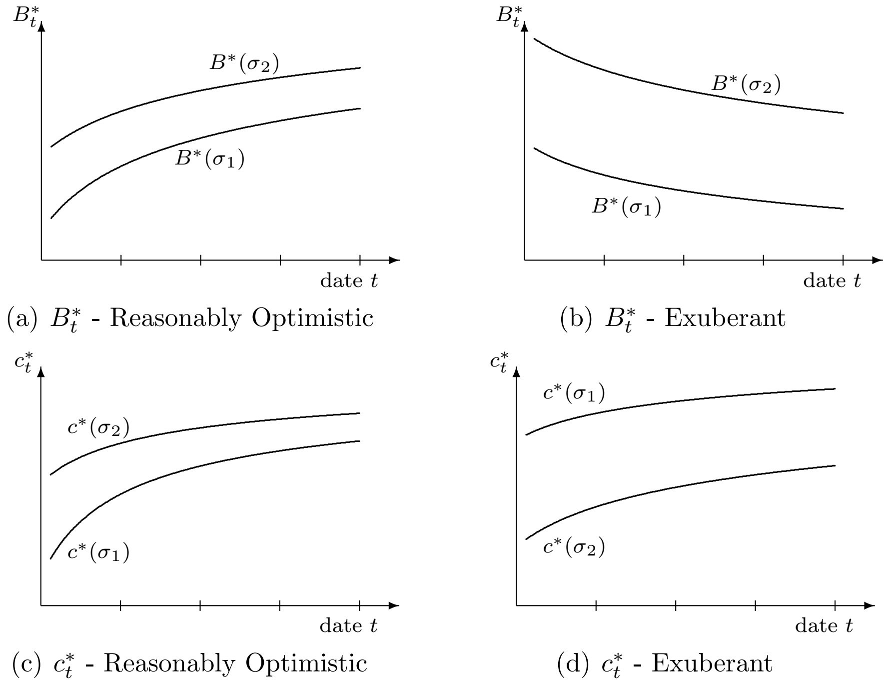
The Effects of Transient Risk and Intrinsic Risk

The following theorem describes the effects of the project's initial transient risk on the equilibrium paths of the incentive intensity, investment and effort.

Theorem 5 (Effects of Transient Risk)
Consider any pair of possible values, $\sigma_1 > \sigma_2$ of the project's initial transient risk σ_0.

- $B_T^*(\sigma_1) \leq B_T^*(\sigma_2)$.
- *If* $B_T^*(\sigma_1) \leq 1 \leq B_T^*(\sigma_2)$ *with at least one strict inequality, then* $B_t^*(\sigma_1) < B_t^*(\sigma_2)$; $\eta_t^*(\sigma_1) < \eta_t^*(\sigma_2)$ *for all* t. $c_t^*(\sigma_1)$ *could be greater than, equal to, or less than* $c_t^*(\sigma_2)$.
- *If* $B_T^*(\sigma_1) \leq B_T^*(\sigma_2) \leq 1$ *then* $B_t^*(\sigma_1) \leq B_t^*(\sigma_2)$; $c_t^*(\sigma_1) \leq c_t^*(\sigma_2)$; $\eta_t^*(\sigma_1) \leq \eta_t^*(\sigma_2)$.
- *If* $1 \leq B_T^*(\sigma_1) \leq B_T^*(\sigma_2)$ *with at least one strict inequality, then there exists* $u \in [0, T)$ *such that* $B_t^*(\sigma_1) > B_t^*(\sigma_2); c_t^*(\sigma_1) < c_t^*(\sigma_2)$; $\eta_t^*(\sigma_1) > \eta_t^*(\sigma_2)$ *for* $t < u$ *and* $B_t^*(\sigma_1) < B_t^*(\sigma_2); c_t^*(\sigma_1) > c_t^*(\sigma_2); \eta_t^*(\sigma_1) < \eta_t^*(\sigma_2)$ *for* $t > u$.

Figure 3: The sensitivity of total incentives (B_t^*) and investment schedules (c_t^*) to transient risk for reasonably optimistic and exuberant agent. Here, σ_n, $n = 1, 2$ are initial levels of transient risk, $\sigma_1 > \sigma_2$. The initial levels of agent optimism for the reasonably optimistic and exuberant agent are Ω_1 and Ω_2, respectively, where $\Omega_1 < \lambda(T\sigma_n^2 + s^2) < \Omega_2$ for $n = 1, 2$



(a) B_t^* - Reasonably Optimistic

(b) B_t^* - Exuberant

(c) c_t^* - Reasonably Optimistic

(d) c_t^* - Exuberant

Figure 3 illustrates the results of Theorem 5 by showing the effects of transient risk on the agent's total incentive intensity path in the scenarios where the agent is initially reasonably optimistic and exuberant.

By (5), an increase in the initial transient risk lowers the degree of agent optimism at the terminal date T. It follows from (46) that $B_T^*(\sigma_1) < B_T^*(\sigma_2)$. If $B_T^*(\sigma_1) \leq 1 \leq B_T^*(\sigma_2)$, the assertions of the theorem immediately follow from Theorem 3. The non-monotonicity of the optimal investment function (see Proposition 1) implies that $c_t^*(\sigma_1)$ could be greater than, equal to, or less than $c_t^*(\sigma_2)$.

Suppose that $B_T^*(\sigma_1) < B_T^*(\sigma_2) \leq 1$. By (3), an increase in the initial transient risk increases the adverse selection costs in (39). If the agent is moderately optimistic so that $B_t^* < 1$, the effects of adverse selection costs dominate the effects of asymmetric beliefs. The increase in adverse selection costs with the initial transient risk, therefore, has a negative effect on total incentives so that the agent's total incentive intensities, effort, and the principal's investments decline with the initial transient risk.

Suppose the agent is initially "exuberant" so that $B_t^* > 1$. In this region, the effects of agent optimism and adverse selection costs are both significant. By (5), the degree of agent optimism, Ω_t, at each date declines with the initial transient risk. The decline in the degree of agent optimism has a negative effect on total incentives in (39). An increase in the initial transient risk also increases adverse selection costs by (3) and (39). In sufficiently early periods of the relationship, it could be optimal to exploit the agent's exuberance by increasing his total incentives and his effort as the initial transient risk increases. The inverted U-shape nature of the optimal investment function in this region (see Proposition 1) implies that it is optimal for the principal to lower her investment so that output is dominated by the agent's effort. In later periods, however, the degree of agent optimism falls below a threshold so that the agent's total incentives and effort decrease with the initial transient risk. The principal's investment increases to offset the decline in the agent's effort.

The results of the theorem show that the complex interplay among agent optimism, adverse selection and risk-sharing costs could cause the relationship between transient risk and incentives to be positive or negative. Further, depending on the level of agent optimism, transient risk could have a positive or negative effect on the principal's investments.

The following theorem describes the effects of the project's intrinsic risk on the equilibrium paths of the incentive intensity, investment and effort.

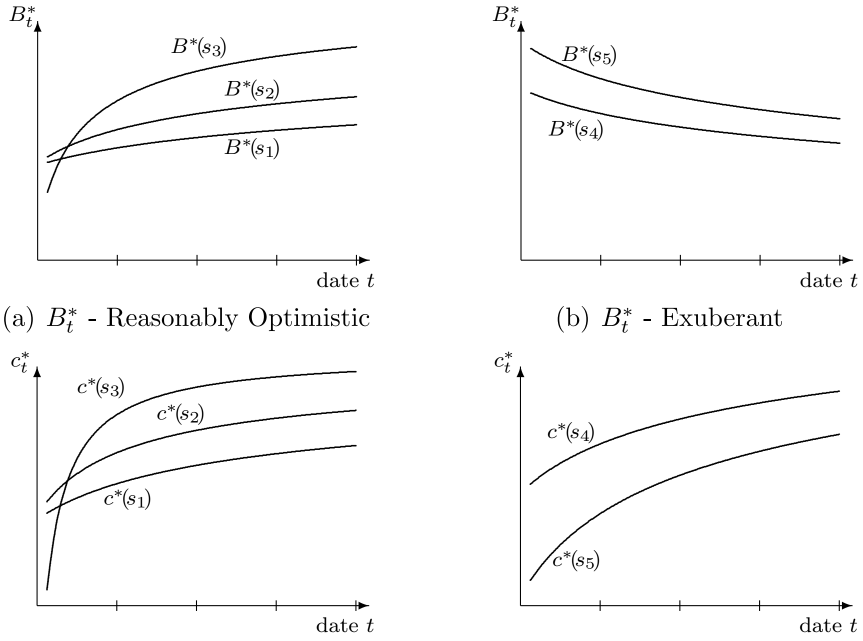
Theorem 6 (The Effects of Intrinsic Risk)

Consider any pair of possible values, $s_1 > s_2$ of the project's intrinsic risk. We could have $B^*_T(s_1) \geq B^*_T(s_2)$ or $B^*_T(s_1) \leq B^*_T(s_2)$.

- *Suppose $B^*_T(s_1) \leq B^*_T(s_2)$:*
  - *If $B^*_T(s_1) \leq 1 \leq B^*_T(s_2)$ then $B^*_t(s_1) < B^*_t(s_2)$; $\eta^*_t(s_1) < \eta^*_t(s_2)$ for all t. $c^*_t(s_1)$ could be greater than, equal to, or less than $c^*_t(s_2)$.*
  - *If $B^*_T(s_1) < B^*_T(s_2) \leq 1$ then there exists $u \in [0, T)$ such that $B^*_t(s_1) > B^*_t(s_2); c^*_t(s_1) > c^*_t(s_2)$; $\eta^*_t(s_1) > \eta^*_t(s_2)$ for $t < u$ and $B^*_t(s_1) < B^*_t(s_2); c^*_t(s_1) < c^*_t(s_2); \eta^*_t(s_1) < \eta^*_t(s_2)$ for $t > u$.*
  - *If $1 \leq B^*_T(s_1) < B^*_T(s_2)$ then (i) $B^*_t(s_1) < B^*_t(s_2)$; (ii) $c^*_t(s_1) > c^*_t(s_2)$; and (iii) $\eta^*_t(s_1) < \eta^*_t(s_2)$ for all t.*
- *Suppose $B^*_T(s_1) > B^*_T(s_2)$*[3]
  - *If $1 \geq B^*_T(s_1) > B^*_T(s_2)$ then $B^*_t(s_1) > B^*_t(s_2); c^*_t(s_1) > c^*_t(s_2)$; $\eta^*_t(s_1) > \eta^*_t(s_2)\ \forall\, t$.*

[3] In this case it is not possible to have $B^*_T(s_1) \geq 1 \geq B^*_T(s_2)$.

Figure 4: The sensitivity of total incentives (B_t^*) and investment schedules (c_t^*) to intrinsic risk for a reasonably optimistic and exuberant agent. Here, s_n, $n = 1, 2, ..., s_5$ are levels of intrinsic risk, $s_1 > s_2 > s_3$ and $s_4 > s_5$. The initial level of optimism for the optimistic (exuberant) agent is less (more) than $\lambda(T\sigma_0^2 + s_n^2)$, n=1,2,3 (n=4,5).



(a) B_t^* - Reasonably Optimistic

(b) B_t^* - Exuberant

(c) c_t^* - Reasonably Optimistic

(d) c_t^* - Exuberant

– *If* $B_T^*(s_1) > B_T^*(s_2) \geq 1$ *then there exists* $u \in [0, T)$ *such that* $B_t^*(s_1) > B_t^*(s_2); c_t^*(s_1) < c_t^*(s_2); \eta_t^*(s_1) > \eta_t^*(s_2)$ *for* $t > u$ *and* $B_t^*(s_1) < B_t^*(s_2); c_t^*(s_1) < c_t^*(s_2); \eta_t^*(s_1) < \eta_t^*(s_2)$ *for* $t < u$.

Figure 4 illustrates the results of the theorem by showing the effects of intrinsic risk on the agent's total incentive intensity path in the scenarios where the agent is initially reasonably optimistic and exuberant.

First, we note that by (5), an increase in the intrinsic risk increases the degree of agent optimism at each date. By (39), however, an increase in the intrinsic risk also increases the costs of risk-sharing. The agent's total incentive intensity at the terminal date T could, therefore, increase or decrease depending on the relative effects of intrinsic risk on these two components of (39).

We discuss the intuition for our results when $B_T^*(s_1) < B_T^*(s_2)$. (The intuition for the results in the case where $B_T^*(s_1) > B_T^*(s_2)$ is analogous and is omitted.) If $B_T^*(s_1) \leq 1 \leq B_T^*(s_2)$, the assertions follow from Theorem 3 and Proposition 1.

If $B_T^*(s_1) < B_T^*(s_2) \leq 1$, then the corresponding total incentive intensity paths are monotonically increasing by Theorem 3. When $T - t$ is below a threshold—that is, in later periods of the relationship—the increase in the costs of risk-sharing in (39) associated with an increase in the intrinsic risk outweighs the positive effects of intrinsic risk on the degree of agent optimism. Hence, the agent's total incentive intensity, the principal's investment and the agents' effort decline. When $T - t$ is above a threshold, however, the increase in the costs of risk-sharing with intrinsic risk is potentially offset by the increase in the degree of agent optimism. By (5), the degree of agent optimism is higher in early periods of the relationship. An increase in the intrinsic risk also has an effect on adverse selection costs that depend on the future total incentive intensities (see 39). Because future total incentive intensities decline with intrinsic risk, the current adverse selection costs decline, which has a positive effect on total incentives. Put differently, the effects of the agent's implicit incentives to influence the principal's learning about the project's quality are much stronger early in the relationship. These implicit incentives potentially become stronger as the intrinsic risk increases. Consequently, the total incentive intensities, the principal's investments and the agent's effort could increase with intrinsic risk in early periods of the relationship.

Suppose now that the total incentive intensities are greater than or equal to one for s_1 and s_2 so that the corresponding total incentive intensity paths are monotonically decreasing by Theorem 3. By (3), an increase in the intrinsic risk increases the project's transient risk at each future date that, in turn, increases the adverse selection costs by (39). This effect coupled with the increase in the costs of risk-sharing cause the total incentive intensities and the agent's effort to decline at each date. The non-monotonicity of the optimal investment function in this region (see Proposition 1) causes the principal's investment to increase with intrinsic risk.

Broadly, Theorems 5 and 6 show that the presence of imperfect information about project quality, asymmetric beliefs, and inter-temporal adverse selection arising from the agent's unobservable effort leads to complex effects of risk on incentives. These results contrast sharply with the predictions of traditional dynamic principal-agent models such as Holmstrom and Milgrom (1987) in which risk unambiguously has a negative effect on incentives. Prendergast (1999), however, highlights the lack of consensus in the empirical literature on the relation between risk and incentives. Our study suggests an alternate explanation for the data by showing that the effects of asymmetric beliefs and implicit incentives could lead to a positive or negative relation between risk and incentives.

We can also characterize the effects of the agent's risk aversion λ on the contractual dynamics. Since its effects are similar to the effects of the project's intrinsic risk as described in the first part of Theorem 6, we avoid stating the precise results.

6.3 Project Characteristics and Firm/Project Value

We now discuss the effects of project characteristics on the project/firm value. By (40) and (41), the firm value H_0 at date 0 is

$$H_0 = E_0^{Pr}\left[\int_0^T (\Theta + \Phi(c_t^*, \eta_t^*))dt\right], \tag{49}$$

where the expectation is with respect to the principal's beliefs.

By (49), the effects of underlying parameters on firm value clearly depend on their effects on the discretionary output rate $\Phi(c_t^*, \eta_t^*)$ As the following proposition shows, the non-monotonic behavior of the optimal investment function causes the discretionary output rate $\Phi(c_t^*, \eta_t^*)$ at any date to also vary non-monotonically with the total incentive intensity B_t^*.

Proposition 5 (Discretionary Output and Total Incentive Intensity)
The discretionary output rate $\Phi(c_t^*, \eta_t^*)$ at any date varies non-monotonically in an inverted U-shaped manner with the agent's total incentive intensity B_t^*. It attains its maximum at $B_t^* = \frac{\gamma}{\beta+\alpha\gamma} > 1$.

The discretionary output in any period is determined by the principal's investment and the agent's effort. Because the investment varies in an inverted U-shaped manner by Proposition 1, the discretionary output also varies in an inverted U-shaped manner. The maximum occurs for a total incentive intensity that exceeds one because the agent's effort (given by 36) depends on the total incentive intensity and the principal's investment. Below a trigger level of the total incentive intensity that exceeds one, the agent's effort increases. Above the trigger level, the decline in the principal's investment dominates so that the agent's effort and the discretionary output decrease. The following theorem examines the relationship between the agent's total incentives and firm value.

Theorem 7 (Total Incentives and Firm Value)
(i) Suppose the agent is initially reasonably optimistic so that his total incentive intensities

B_t^* are less than one and increase over time as described in Theorem 3. An increase in the agent's total incentive intensity path increases firm value.
(ii) Suppose the agent is initially exuberant so that his total incentive intensities B_t^* are greater than one and decrease over time as described in Theorem 3.

- *If $B_0^* < \frac{\gamma}{\beta+\alpha\gamma}$, then an increase in the total incentive intensity path increases firm value.*
- *If $B_0^* \geq \frac{\gamma}{\beta+\alpha\gamma}$, then an increase in the total incentive intensity path could cause firm value to increase or decrease at date t.*

By Proposition 5, the discretionary output increases with the total incentive intensity when the latter is less than one. By Theorem 3, the agent's total incentive intensities are less than one if the agent is initially reasonably optimistic. An increase in the total incentive intensity path, therefore, increases the discretionary output at each date and, consequently, the firm's value. By Proposition 5, if the agent is exuberant, the discretionary output at any date declines with the total incentive intensity if the latter is greater than $\frac{\gamma}{\beta+\alpha\gamma}$, but increases with the total incentive intensity if the latter is less than $\frac{\gamma}{\beta+\alpha\gamma}$. It follows that, depending on the level of agent optimism, and the duration of the relationship, firm value could increase or decrease with the total incentive intensity path.

By Theorems 4 and 7, if the agent is reasonably optimistic, an increase in the degree of optimism increases the agent's total incentive intensities at each date and, therefore, the firm's values at each date. If the agent is exuberant, however, the firm's value could increase or decrease with the agent's optimism. The project's intrinsic and transient risks have much more complex effects on firm value because they have non-monotonic effects on the agent's total incentive intensity path.

The above results have interesting implications for the empirical literature that investigates the relation between managerial incentives and firm value. Himmelberg et al (1999) note that previous empirical studies find non-monotonic relationships between manager ownership and firm value. They point out that cross-section empirical analyses do not appropriately control for unobserved sources of firm heterogeneity that endogenously determine managerial incentives and firm value. Consistent with empirical evidence, our results imply that, in the cross-section, firm value varies non-monotonically with managerial incentives. Moreover, our analysis identifies key determinants of firm heterogeneity—the degree of asymmetry in beliefs, intrinsic risk and transient risk—that lead to the non-monotonic variation of firm value.

7 Conclusions

We develop a continuous-time principal–agent model to study how uncertainty and asymmetric beliefs, asymmetric risk attitudes, two-sided actions, and inter-temporal adverse selection interact to affect dynamic contracts. We show that equilibrium contracts are determined by a solution to a first order nonlinear ODE. We exploit the properties of the ODE to derive a number of novel implications that demonstrate how asymmetric beliefs, agency conflicts, and dynamic adverse selection interact to affect optimal contracts. In particular, our results show that asymmetric beliefs play a central role in potentially reconciling empirical findings such as the "private equity" puzzle, the tenuous relation between risk and incentives, the non-monotonic relation between firm value and incentives. We also derive potentially testable implications for the effects of permanent and transient components of risk on compensation and investment schedules.

From an empirical standpoint, one could apply the framework to study settings such as venture capital (VC) and R&D where asymmetric beliefs likely play an important role. We could estimate the parameters of our structural model to VC or R&D project data and obtain quantitative assessments of the impacts of asymmetric beliefs, adverse selection, and risk-sharing on optimal contracts.

Appendix

Proof of Theorem 1

The proof proceeds by backward induction. It is easy to extend the analysis of the special case in Section 4.1 to show that the assertions of the theorem hold in the last period. Consider any date $t \leq T - \Delta t$. Assume that the assertions of the theorem are true over the interval $[t + \Delta t, T - \Delta t]$. In addition, suppose that the agent truthfully announces his type at each date and state (on or off the equilibrium path) in the interval $[t + \Delta t, T - \Delta t]$, and that the assertions of the theorem are also true at "off-equilibrium nodes" at which the principal's inferences of the agent's past effort choices differ from his actual effort choices. In particular, at any off-equilibrium node at date $u \in [t + \Delta t, T - \Delta t]$, the agent's total incentive intensity solves (32) with the degree of agent optimism equal to Ω_u^{Ag}; the true degree of agent optimism at that node.

Suppose that the contractual parameters in period $[t, t + \Delta t]$ are (a, b) and the principal's investment is c. Suppose the agent exerts effort η in period $[t, t + \Delta t]$. Using the inductive assumptions, we can show that the agent's actual continuation utility ratio (8) is

$$
\begin{aligned}
CUR_t^{Ag} &= E_t^{Ag}\left[\exp\left(-\lambda\left(a\Delta t+b(V_{t+\Delta t}-V_t)-k\eta^\gamma\Delta t+\sum_{u=t+\Delta t}^{T-\Delta t}[a_u^*\Delta t+b_u^*\Delta V_u-k\left(\eta_u^*\right)^\gamma\Delta t]\right)\right)\right] \\
&= E_t^{Ag}\left[\exp\left(-\lambda\left(\begin{array}{c}\left(a+b(\Theta+Ac^\alpha\eta^\beta)-k\eta^\gamma\right)\Delta t+bs\Delta W_t \\ +\sum\limits_{u=t+\Delta t}^{T-\Delta t}\left[a_u^*+b_u^*\left(\Theta+A\left(c_u^*\right)^\alpha\left(\eta_u^*\right)^\beta\right)-k\left(\eta_u^*\right)^\gamma]\Delta t+b_u^*s\Delta W_u\right]\end{array}\right)\right)\right] \\
&= E_t^{Ag}\left[\exp\left(-\lambda\left(\begin{array}{c}\left(a+b(\Theta+Ac^\alpha\eta^\beta)-k\eta^\gamma\right)\Delta t+bs\Delta W_t \\ +\sum\limits_{u=t+\Delta t}^{T-\Delta t}\left[\begin{array}{c}\left[(1-b_u^*)\left(A(c_u^*)^\alpha(\eta_u^*)^\beta+\mu_u^{Pr}\right)-c_u^*\right]\Delta t \\ +\left[b_u^*\left(\Theta+A\left(c_u^*\right)^\alpha\left(\eta_u^*\right)^\beta\right)-k\left(\eta_u^*\right)^\gamma\right]\Delta t+b_u^*s\Delta W_u\end{array}\right]\end{array}\right)\right)\right].
\end{aligned}
\tag{50}
$$

By (1) and (4)

$$
\begin{aligned}
\mu_u^{Pr} &= \frac{s^2\mu_t^{Pr}+\sigma_t^2\sum\limits_{v=t}^{u-\Delta t}\left(\Theta\Delta t+s\Delta W_v+(\Phi(c_v,\eta_v^{Ag})-\Phi(c_v,\eta_v^{Pr}))\Delta t\right)}{s^2+(u-t)\sigma_t^2} \\
&= \frac{s^2\mu_t^{Pr}+\sigma_t^2\left(\Theta\Delta t+s\Delta W_t+\left(\Phi(c_t,\eta)-\Phi(c_t,\eta_t^{Pr})\right)\Delta t\right)}{s^2+(u-t)\sigma_t^2}+\frac{\sigma_t^2\sum\limits_{v=t+\Delta t}^{u-\Delta t}(\Theta\Delta t+s\Delta W_v)}{s^2+(u-t)\sigma_t^2}.
\end{aligned}
\tag{51}
$$

where the last equality follows from the inductive assumption that the agent's effort choices are incentive compatible for $v\geq t+\Delta t$, and the hypothesis that the agent exerts effort η over the period $[t,t+\Delta t]$. Substituting (51) in (50), and using the fact that (b_v^*,c_v^*,η_v^*) are all $\mathcal{F}_t^{Pr}-$measurable for $v\geq t+\Delta t$ by (51) and the inductive assumptions, the agent's implementable effort is

$$
\begin{aligned}
\eta(B,c) &= \arg\max_\eta\left(b+\sum_{v=t+\Delta t}^{T-\Delta t}(1-b_v^*)\frac{\sigma_t^2}{\sigma_t^2(v-t)+s^2}\Delta t\right)Ac^\alpha\eta^\beta-k\eta^\gamma \\
&= \arg\max_\eta BAc^\alpha\eta^\beta-k\eta^\gamma=\left(\frac{A\beta c^\alpha B}{\gamma k}\right)^{\frac{1}{\gamma-\beta}}.
\end{aligned}
\tag{52}
$$

The principal's inference of the agent's effort over the period $[t,t+\Delta t]$, $\eta_t^{Pr}=\eta(B,c)$.

Suppose that the principal's inferences of the agent's prior effort choices are $\{\eta_u^{Pr};u\leq t-\Delta t\}$ and the agent announces that his effort choices prior to date t are $\{\widehat{\eta}_u^{Ag};u\leq t-\Delta t\}$. As in our analysis of the special case in Section 4.1, we conjecture that the principal does not alter her inferences of the agent's prior effort choices. The contract must, therefore, satisfy the promise keeping constraint (7) for the principal, where the principal's current mean assessment of the project's quality is μ_t^{Pr}. It then follows that $a=a(b,c)$ given by

$$
a(b,c)=(1-b)\left[Ac^\alpha\eta(b,c)^\beta+\mu_t^{Pr}\right]-c. \tag{53}
$$

The agent's current mean assessment of the project's quality based on his "announced" past effort choices is $\widehat{\mu}_t^{Ag}$. Substituting (52) and (53) in (50), the agent's continuation utility ratio based on his “announced” effort choices is $\widehat{CUR}_t^{Ag}(b,c)$

$$=\widehat{E}_t^{Ag}\left[\exp\left(-\lambda\left(\begin{array}{c}\left[(1-b)\left[Ac^{\alpha}\eta(B,c)^{\beta}+\mu_t^{Pr}\right]-c+b(\Theta+Ac^{\alpha}\eta(B,c)^{\beta})-k\eta(B,c)^{\gamma}\right]\Delta t+bs\Delta W_t\\ +\sum\limits_{u=t+\Delta t}^{T-\Delta t}\left[\begin{array}{c}\left[(1-b_u^*)\left(A(c_u^*)^{\alpha}(\eta_u^*)^{\beta}+\mu_u^{Pr}\right)-c_u^*\right]\Delta t\\ +\left[b_u^*\left(\Theta+A\left(c_u^*\right)^{\alpha}\left(\eta_u^*\right)^{\beta}\right)-k\left(\eta_u^*\right)^{\gamma}\right]\Delta t+b_u^*s\Delta W_u\end{array}\right]\end{array}\right)\right)\right]$$

$$=\widehat{E}_t^{Ag}\left[\exp\left(-\lambda\left(\begin{array}{c}\left[(1-b)\mu_t^{Pr}-c+b\Theta+Ac^{\alpha}\eta(B,c)^{\beta}-k\eta(B,c)^{\gamma}\right]\Delta t+bs\Delta W_t\\ +\sum\limits_{u=t+\Delta t}^{T-\Delta t}\left[\frac{\gamma-\beta-\alpha\gamma}{\alpha\gamma}c_u^*\Delta t+b_u^*\left(\Theta\Delta t+s\Delta W_u\right)\right]\\ +\sum\limits_{u=t+\Delta t}^{T-\Delta t}(1-b_u^*)\left[\frac{s^2\mu_t^{Pr}\Delta t}{s^2+(u-t)\sigma_t^2}+\frac{\sigma_t^2\Delta t\sum_{v=t}^{u-\Delta t}(\Theta\Delta t+s\Delta W_v)}{s^2+(u-t)\sigma_t^2}\right]\end{array}\right)\right)\right] \tag{54}$$

where $\widehat{E}_t^{Ag}[\cdot]$ denotes the conditional expectation under the agent’s “announced” beliefs so that $\widehat{E}_t^{Ag}[\Theta]=\widehat{\mu}_t^{Ag}$. The second equality above follows by substituting (51) (setting $\eta=\eta_t^{Pr}=\eta(B,c)$) and the fact that $A(c_u^*)^{\alpha}(\eta_u^*)^{\beta}-c_u^*-k(\eta_u^*)^{\gamma}=\frac{\gamma-\beta-\alpha\gamma}{\alpha\gamma}c_u^*$, by the inductive assumptions. Changing the order of the summations in the third term of (54) we have $\widehat{CUR}_t^{Ag}(b,c)$

$$=\quad\widehat{E}_t^{Ag}\left[\exp\left(-\lambda\left(\begin{array}{c}\left[(1-b)\mu_t^{Pr}-c+Ac^{\alpha}\eta(B,c)^{\beta}-k\eta(B,c)^{\gamma}\right]\Delta t+b(\Theta\Delta t+s\Delta W_t)\\ +\sum\limits_{u=t+\Delta t}^{T-\Delta t}\left[\frac{\gamma-\beta-\alpha\gamma}{\alpha\gamma}c_u^*\Delta t+b_u^*\left(\Theta\Delta t+s\Delta W_u\right)\right]\\ +\sum\limits_{u=t+\Delta t}^{T-\Delta t}\left[(1-b_u^*)\frac{s^2\mu_t^{Pr}\Delta t}{s^2+(u-t)\sigma_t^2}+\sigma_t^2\Delta t\sum\limits_{v=u+\Delta t}^{T-\Delta t}(1-b_v^*)\frac{\Theta\Delta t+s\Delta W_u}{s^2+(v-t)\sigma_t^2}\right]\end{array}\right)\right)\right] \tag{55}$$

Re-arranging terms we have $\widehat{CUR}_t^{Ag}(b,c)$

$$=\widehat{E}_t^{Ag}\left[\exp\left(-\lambda\left(\begin{array}{c}\left[(1-b)\mu_t^{Pr}-c+Ac^{\alpha}\eta(B,c)^{\beta}-k\eta(B,c)^{\gamma}\right]\Delta t+B\left(\Theta\Delta t+s\Delta W_t\right)\\ +\sum\limits_{u=t+\Delta t}^{T-\Delta t}\left[\frac{\gamma-\beta-\alpha\gamma}{\alpha\gamma}c_u^*\Delta t+(1-b_u^*)\frac{s^2\mu_t^{Pr}}{s^2+(u-t)\sigma_t^2}\Delta t+B_u^*\left(\Theta\Delta t+s\Delta W_u\right)\right]\end{array}\right)\right)\right] \tag{56}$$

where

$$B_u^*=\left(b_u^*+\sum_{v=u+\Delta t}^{T-\Delta t}(1-b_v^*)\frac{\sigma_t^2}{\sigma_t^2(v-t)+s^2}\Delta t\right). \tag{57}$$

Similar to our analysis in Section 4.1, we conjecture that the principal’s investment and the agent’s incentive intensity minimize the agent’s “announced” continuation utility ratio. Using arguments similar to those in the proof of Theorem 3.1 in Giat et al (2010), we can show that the optimal investment as a function of the incentive intensity, $c(B)$, is given by (33). The agent’s optimal total incentive intensity solves

$$B_t^*=\arg\max_B\widehat{\Omega}_t^{Ag}B-0.5\lambda\left(\sigma_t^2\left[B+\sum_{u=t+\Delta t}B_u^*\right]^2\Delta t+s^2\left[B^2+\sum_{u=t+\Delta t}^{T-\Delta t}(B_u^*)^2\right]\right)+\frac{\gamma-\beta-\alpha\gamma}{\alpha\gamma}c(B). \tag{58}$$

Because the contractual parameters are chosen to maximize the “apparent” or “announced” surplus based on the agent’s announced effort choices, it is incentive compatible for the agent to tell the truth so that the contractual parameters maximize the "actual" surplus. Therefore, $\widehat{\Omega}_t^{Ag} = \Omega_t^{Ag}$. Since $\{B_u^*; u > t\}$ do not depend on B, the agent's optimal total incentive intensity solves (32).

Further, note that, along the equilibrium path, the contract as constructed above maximizes the agent's expected utility conditional on delivering the principal her promised payoff. As described above, the principal correctly infers the agent's effort in equilibrium. Hence, our conjecture that the principal does not alter her inferences about the agent's past effort in response to his announcements is correct. This completes the inductive step in the analysis.

To complete the proof of the theorem, we now establish the second equality of (34) by induction. This equality holds for $b_{T-\Delta t}^*$. Assume now that it is true for $t+\Delta t$, i.e.

$$b_{t+\Delta t}^* = B_{t+\Delta t}^* - \sum_{v=t+2\Delta t}^{T-\Delta t} (1 - b_v^*) \frac{\sigma_0^2}{\sigma_0^2 v + s^2} \Delta t = B_{t+\Delta t}^* - \frac{\sigma_0^2}{\sigma_0^2 (t + 2\Delta t) + s^2} \Delta t \sum_{v=t+2\Delta t}^{T-\Delta t} (1 - B_v^*)$$

$$1 - b_{t+\Delta t}^* = 1 - B_{t+\Delta t}^* + \frac{\sigma_0^2}{\sigma_0^2 (t + 2\Delta t) + s^2} \Delta t \sum_{v=t+2\Delta t}^{T-\Delta t} (1 - B_v^*). \tag{59}$$

We now consider date t. By definition

$$\begin{aligned}
b_t^* &= B_t^* - \sum_{v=t+\Delta t}^{T-\Delta t} (1 - b_v^*) \frac{\sigma_0^2 \Delta t}{\sigma_0^2 v + s^2} \\
&= B_t^* - (1 - b_{t+\Delta t}^*) \frac{\sigma_0^2 \Delta t}{\sigma_0^2 (t+\Delta t) + s^2} - \frac{\sigma_0^2 \Delta t}{\sigma_0^2 (t+2\Delta t) + s^2} \sum_{v=t+2\Delta t}^{T-\Delta t} (1 - B_v^*) \\
&= B_t^* - \Big(1 - B_{t+\Delta t}^* + \frac{\sigma_0^2 \Delta t}{\sigma_0^2 (t+2\Delta t) + s^2} \sum_{v=t+2\Delta t}^{T-\Delta t} (1 - B_v^*)\Big) \frac{\sigma_0^2 \Delta t}{\sigma_0^2 (t+\Delta t) + s^2} - \frac{\sigma_0^2 \Delta t}{\sigma_0^2 (t+2\Delta t) + s^2} \sum_{v=t+2\Delta t}^{T-\Delta t} (1 - B_v^*) \\
&= B_t^* - \frac{\sigma_0^2}{\sigma_0^2 (t+\Delta t) + s^2} \Delta t \sum_{v=t+\Delta t}^{T-\Delta t} (1 - B_v^*),
\end{aligned}$$

where the third equality follows by (59). ∎

Proof of Proposition 2

By Proposition 1, $c'(0) = \infty$. It follows from (39) that $F_t'(0) = \infty$. By (32), $B_t^* > 0$.

Suppose that $\frac{\Omega_0}{\lambda s^2} < 1$. It follows from (5) that $\frac{\Omega_t}{\lambda s^2} < 1$ for $t > 0$. Hence, $\Omega_t - \lambda s^2 B < 0$ for $B > 1$. Because $B_t^* > 0$ for all t, and $c'(B) < 0$ for $B > 1$ by Proposition 1, it follows that

$$F_t'(B) = \Omega_t - \lambda s^2 B - \lambda \sigma_t^2 \left[B + \sum_{u=t+\Delta t}^{T-\Delta t} B_u^* \right] \Delta t + \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma} c'(B) < 0$$

for $B > 1$. Therefore, by (32), $B_t^* \leq 1$.

Suppose now that $1 \leq \frac{\Omega_0}{\lambda s^2} < B_M$. If $B > \frac{\Omega_0}{\lambda s^2}$ then $\Omega_t - \lambda s^2 B < 0$ so that

$$F_t'(B) = \Omega_t - \lambda s^2 B - \lambda \sigma_t^2 \left[B + \sum_{u=t+\Delta t}^{T-\Delta t} B_u^* \right] \Delta t + \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma} c'(B) < 0.$$

It follows that we must have $B_t^* \leq \frac{\Omega_0}{\lambda s^2}$. ∎

Proof of Proposition 3

The proof requires a precise interpretation of equation (41), which describes the evolution of the incremental payoff. We consider the process $V(\cdot)$ to be a *given* random process on a probability space with investment and effort altering its probability distribution.

We consider an underlying probability space (Ω, F) with probability measures Q^ℓ, $\ell \in \{Pr, Ag\}$, representing the principal's and agent's beliefs. Θ is a normal random variable with mean μ_0^ℓ and variance σ_0^2 under measure Q^ℓ and $\widehat{W}$ is a standard Brownian motion. The complete and augmented filtration of the probability space generated by the Brownian motion $\widehat{B}(\cdot)$ is denoted by $\{F_t\}$. Consider the process $V(\cdot) = s\widehat{W}(\cdot)$ where s^2 is the intrinsic risk of the project. We use the Girsanov transformation (see Oksendal, 2003) to obtain new probability measures on (Ω, F) such that the process $V(\cdot)$ evolves as in (41).

Suppose that $\eta(\cdot)$ and $c(\cdot)$ are strictly positive, square-integrable $\{F_t\}$-progressively measurable stochastic processes (under the measures Q^{Pr} and Q^{Ag}) defined on the time horizon $[0, T]$ describing the agent's effort and the principal's investments over time. Define

$$\zeta_{c,\eta}(t) = \exp[\int_0^t (\Theta + Ac(u)^\alpha \eta(u)^\beta) s^{-1} d\widehat{W}(u) - \frac{1}{2}\int_0^t (\Theta + Ac(u)^\alpha \eta(u)^\beta)^2 s^{-2} du]$$

$$W_{c,\eta}(t) = \widehat{W}(t) - \int_0^t (\Theta + Ac(u)^\alpha \eta(u)^\beta) s^{-1} du \ .$$

The process $\zeta_{c,\eta}(\cdot)$ is a positive, square-integrable martingale.[4] Define the new measure $\Pi_{c,\eta}^\ell; \ell \in \{Pr, Ag\}$

$$\frac{d\Pi_{c,\eta}^\ell}{dQ^\ell} = \zeta_{c,\eta}(T).$$

By Girsanov's theorem (see Oksendal, 2003), the process $W_{c,\eta}(\cdot)$ is a Brownian motion under the measure $\Pi_{c,\eta}^\ell$. Further, under this measure, the process $V(\cdot)$ evolves as

$$dV(t) = [\Theta + Ac(t)^\alpha \eta(t)^\beta]dt + sdW_{c,\eta}(t) \ . \tag{60}$$

Equation (60) describes the evolution of the incremental payoff process and is identical to equation (41). However, the Brownian motion and the probability measures representing the principal's and agent's beliefs depend on the investment and effort processes. It is important to keep in mind that $V(\cdot)$ is a *fixed* process whose sample paths are not affected by investment and effort. Investment

[4]The processes are assumed to satisfy the Novikov condition $E^\ell \exp[\frac{1}{2}\int_0^T (\Theta + Ac(u)^\alpha \eta(u)^\beta)^2 s^{-2} du] < \infty$, $\ell \in \{Pr, Ag\}$. Note that, because the *equilibrium* investment and effort processes described in Theorem 1 are *deterministic* and Θ is a normal random variable, the Novikov condition is satisfied by these processes.

and effort, however, alter the probability distribution of the sample paths of $V(\cdot)$. The process

$$dW_{c,\eta}(t) = s^{-1}[dV(t) - (Ac(t)^{\alpha}\eta(t)^{\beta})dt - \mu_t^{\ell}dt] \tag{61}$$

is an $\{F_t\}$-Brownian motion with respect to the probability measure $\Pi_{c,\eta}^{\ell}$. Moreover, the complete and augmented filtration generated by this Brownian motion is $\{F_t\}$. The agent's and principal's mean assessments of project quality Θ at date t, μ_t^{Ag}, μ_t^{Pr} are given by (4).

Let (Q^*, c^*, η^*) denote an optimal contract. It follows from (7) that

$$Q_t^* = E_t^{Pr}\left\{Q_T^* - \int_t^T c_u^* du\right\}.$$

From the above, the process

$$Z_t^* = Q_t^* + \int_0^t c_u^* du \tag{62}$$

is an $\{F_t\}-$martingale under the measure Π_{c^*,η^*}^{Pr}. Because W_{c^*,η^*} is an $\{F_t\}$-Brownian motion under the measure Π_{c^*,η^*}^{Pr} that generates the filtration $\{F_t\}$, it follows from the martingale representation theorem that there exists an $\{F_t\}-$progressively measurable, square-integrable process z_t^* such that

$$\begin{aligned} Z_t^* &= Z_0^* + \int_0^t z_u^* dW_{c^*,\eta^*}(u) \\ &= Q_0 + \int_0^t z_u^* s^{-1}[dV(t) - (Ac(t)^{\alpha}\eta(t)^{\beta})dt - \mu_t^{\ell}dt] \end{aligned} \tag{63}$$

The second equality above follows from (61), (62) and the fact that $Z_0^* = Q_0^* = Q_0$. By (62) and (63),

$$dQ_t^* = -a_t^* dt + (1 - b_t^*)dV_t,$$

where a_t^* is $\{F_t\}-$progressively measurable, and b_t^* is $\{F_t\}-$progressively measurable and square-integrable. It follows that the principal's promised payoff process under the optimal contract evolves as in (42). Since the agent's stake is $V_t - Q_t^*$, it evolves as in (43). ∎

Proof of Theorem 2

We establish that the optimal contract *within the class of affine contracts* in the discrete-time model converges to the optimal contract among the class of *all incentive feasible contracts* in the continuous-time model as the time interval Δt between successive dates tends to zero. Define the processes $b_{\Delta t}^*(\cdot), B_{\Delta t}^*(\cdot), c_{\Delta t}^*(\cdot), \eta_{\Delta t}^*(\cdot), \sigma_{\Delta t}(\cdot)$ in continuous-time by "piecewise constant interpolation" of the corresponding discrete-time processes as follows:

$$\begin{aligned} b_{\Delta t}^*(t) &= b_u^* 1_{(u,u+\Delta t]}(t);\ B_{\Delta t}^*(t) = B_u^* 1_{(u,u+\Delta t]}(t);\ a_{\Delta t}^*(t) = a_u^* 1_{(u,u+\Delta t]}(t) \\ c_{\Delta t}^*(t) &= c_u^* 1_{(u,u+\Delta t]}(t);\ \eta_{\Delta t}^*(t) = \eta_u^* 1_{(u,u+\Delta t]}(t), \end{aligned} \tag{64}$$

where $b_u^*, B_u^*, a_u^*, c_u^*$, and η_u^* are defined in the statement of Theorem 1. Note here that the subscripts in the notation for the processes on the left hand sides of the equalities above denote the length of each interval in the discrete-time model. We require four intermediate lemmas.

Lemma 1

The processes $b^*_{\Delta t}(\cdot), B^*_{\Delta t}(\cdot), a^*_{\Delta t}(\cdot), c^*_{\Delta t}(\cdot), \eta^*_{\Delta t}(\cdot)$ are uniformly bounded for all Δt.

Proof. By Proposition 2, it immediately follows that the process $B^*_{\Delta t}(\cdot)$ is uniformly bounded for all Δt. By the second equality in (34), the process $b^*_{\Delta t}(\cdot)$ is also uniformly bounded. It follows from (33), (36) and (37) that the processes $a^*_{\Delta t}(\cdot), c^*_{\Delta t}(\cdot), \eta^*_{\Delta t}(\cdot)$ are also uniformly bounded for all Δt. Q.E.D.

Lemma 2

The processes $b^*_{\Delta t}(\cdot), B^*_{\Delta t}(\cdot), a^*_{\Delta t}(\cdot), c^*_{\Delta t}(\cdot), \eta^*_{\Delta t}(\cdot)$ converge point-wise to the processes $b^*(\cdot), B^*(\cdot), a^*(\cdot), c^*(\cdot), \eta^*(\cdot)$ defined in the statement of the theorem as $\Delta t \longrightarrow 0$.

Proof. Consider any countable decreasing sequence of subinterval lengths $(\Delta t_1, \Delta t_2, ...)$ where $\Delta t_i \longrightarrow 0$, and the sequence of processes $B^*_{\Delta t_i}(\cdot)$. Because the processes are uniformly bounded by the result of Lemma 1, there exists a subsequence (again denoted by $B^*_{\Delta t_i}(\cdot)$ for notational convenience) that converges pointwise to a process $\widetilde{B}^*$. For $u \in \{0, \Delta t_i, 2\Delta t_i, ..., T - \Delta t_i\}$, it follows from (32) that B^*_u must satisfy the first order condition

$$\Omega_u - \lambda s^2 B^*_u - \lambda \sigma_u^2 \left(\sum_{v=u}^{T-\Delta t_i} B^*_u \right) \Delta t + \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma} c'(B^*_u) = 0$$

It follows from the above, the uniform boundedness of the processes $B^*_{\Delta t}(\cdot)$ for all Δt, and the dominated convergence theorem that the process $\widetilde{B}^*$ must satisfy the following integral equation:

$$\Omega_t - \lambda s^2 \widetilde{B}^*(t) - \lambda \sigma_t^2 \int_t^T \widetilde{B}^*(s) ds + \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma} c'(\widetilde{B}^*(t)) = 0 \qquad \text{or} \qquad (65)$$

$$\frac{\Omega_t}{\lambda \sigma_t^2} - \frac{s^2}{\sigma_t^2} \widetilde{B}^*(t) + \frac{\gamma - \beta - \alpha\gamma}{\lambda\alpha\gamma\sigma_t^2} c'(\widetilde{B}^*(t)) = \int_t^T \widetilde{B}^*(s) ds.$$

By (4) and (3), the above can be re-expressed as

$$\frac{\Omega_0}{\lambda \sigma_0^2} - \frac{s^2 + t\sigma_0^2}{\sigma_0^2} \widetilde{B}^*(t) + \frac{(\gamma - \beta - \alpha\gamma)(s^2 + t\sigma_0^2)}{\lambda\alpha\gamma s^2 \sigma_0^2} c'(\widetilde{B}^*(t)) = \int_t^T \widetilde{B}^*(s) ds. \qquad (66)$$

Differentiating the above with respect to time, we have

$$\left[\frac{(\gamma - \beta - \alpha\gamma)(s^2 + t\sigma_0^2)}{\lambda\alpha\gamma s^2 \sigma_0^2} c''(\widetilde{B}^*(t)) - \frac{s^2 + t\sigma_0^2}{\sigma_0^2} \right] \frac{d\widetilde{B}^*(t)}{dt} + \frac{(\gamma - \beta - \alpha\gamma)}{\lambda\alpha\gamma s^2} c'(\widetilde{B}^*(t)) = 0, \qquad (67)$$

which is identical to (45). Furthermore, it follows directly from (65) that $\widetilde{B}^*(T)$ must satisfy the boundary condition (46).

We now prove that the solution to the ODE (45) and boundary condition (46) is unique. Because the processes $B^*_{\Delta t}(\cdot)$ are uniformly bounded for all Δt, the limit process $\widetilde{B}^*(\cdot)$ constructed above is bounded. It follows that the right hand side of (66) is uniformly bounded for all t so that the left hand side must also be uniformly bounded. Because $\lim_{x \longrightarrow 0} c'(x) = \infty$, the process $\widetilde{B}^*(\cdot)$ must be uniformly bounded below away from zero. Denoting the lower bound by $\underline{B}$, the process $\widetilde{B}^*(\cdot)$ must lie in the interval $[\underline{B}, B_M]$. Rewrite (67) as

$$\frac{d\widetilde{B}^*(t)}{dt} = -\frac{\sigma_0^2}{s^2 + t\sigma_0^2} \frac{c'(\widetilde{B}^*(t))}{\left(c''(\widetilde{B}^*(t)) - \frac{\alpha\gamma\lambda s^2}{\gamma-\beta-\alpha\gamma}\right)} = f(t, \widetilde{B}^*(t)), \tag{68}$$

The derivative of $f(t, B)$ is bounded for $B \in [\underline{B}, B_M]$ so that $f(t, B)$ is uniformly Lipschitz for $B \in [\underline{B}, B_M]$. By the Picard-Lindelof theorem, the solution of the ODE (68) is unique.

We now complete the proof of the lemma by contradiction. Suppose the process $B^*_{\Delta t}(\cdot)$ does not converge point-wise to the process $B^*(\cdot)$. There exists a sequence of processes $\left\{B^*_{\Delta t_i}(\cdot)\right\}$ with $\Delta t_i \longrightarrow 0$ that does not converge to $B^*(\cdot)$ at some t. By the above arguments, there exists a convergent subsequence that converges to a limit process that satisfies the ODE (67). Since the solution to the ODE is unique, the process must be $B^*(\cdot)$, which is a contradiction. The remaining statements of the lemma follow from (34), (35), (36) and (37). Q.E.D.

Lemma 3
The agent's expected utility from the contract $(a^*_{\Delta t}(\cdot), b^*_{\Delta t}(\cdot), c^*_{\Delta t}(\cdot), \eta^*_{\Delta t}(\cdot))$ *converges to his expected utility from the contract* $(a^*(\cdot), b^*(\cdot), c^*(\cdot), \eta^*(\cdot))$ *as* $\Delta t \longrightarrow 0$.

Proof. By (6), (43), and the analysis in the proof of Theorem 1, the agent's expected utility from the contract $(a^*_{\Delta t}(\cdot), b^*_{\Delta t}(\cdot), c^*_{\Delta t}(\cdot), \eta^*_{\Delta t}(\cdot))$ is

$$-E_0^{Ag}\left\{\exp\left(-\lambda\left[P(0) + \int_0^T a^*_{\Delta t}(t)dt + b^*_{\Delta t}(t)dW(t) - \int_0^T k\eta^*_{\Delta t}(t)^\gamma dt\right]\right)\right\} \tag{69}$$

$$= -E_0^{Ag}\exp\left[-\lambda\begin{pmatrix} P(0)+\int_0^T\left(\Omega_{\Delta t}(t)b^*_{\Delta t}(t) - 0.5\lambda s^2\int_0^T B^*_{\Delta t}(t)^2dt + \frac{\gamma-\beta-\alpha\gamma}{\alpha\gamma}c^*_{\Delta t}(t) + \mu^{Pr}_{\Delta t}(t)\right)dt \\ -0.5\lambda\sigma_0^2\left(\int_0^T B^*_{\Delta t}(t)dt\right)^2\end{pmatrix}\right],$$

where $\qquad \sigma_{\Delta t}(\cdot) = \sigma_u 1_{(u,u+\Delta t]}(t) \quad ; \quad \mu^{Pr}_{\Delta t}(t) = \mu_u 1_{(u,u+\Delta t]}(t) \quad ; \quad \Omega_{\Delta t}(t) = \Omega_u 1_{(u,u+\Delta t]}(t).$

By Lemma 2, the expression

$$\Omega_{\Delta t}(t)b^*_{\Delta t}(t) - 0.5\lambda s^2\int_0^T B^*_{\Delta t}(t)^2dt - 0.5\lambda\sigma_0^2\left(\int_0^T B^*_{\Delta t}(t)dt\right)^2 + \frac{\gamma-\beta-\alpha\gamma}{\alpha\gamma}c^*_{\Delta t}(t)$$

is uniformly bounded for all t and for all Δt. It follows from (69) that it suffices to show that the set of random variables $\exp\left[\int_0^T \mu^{Pr}_{\Delta t}(t)dt\right]$ is uniformly integrable. To show this, it suffices to show that

$$E\exp\left[\xi\int_0^T \mu^{Pr}_{\Delta t}(t)dt\right] = E\exp\left[\xi\sum_{u=0}^{T-\Delta t}\mu^{Pr}_u\Delta t\right] < \infty$$

for some $\xi > 1$. The above follows from the fact that μ^{Pr}_u is normally distributed for each u by (4) and the fact that the optimal investment and effort are deterministic. Q.E.D.

The following lemma completes the proof of the theorem by establishing the optimality of the contract $(a^*(\cdot), b^*(\cdot), c^*(\cdot), \eta^*(\cdot))$ in the continuous-time model.

Lemma 4
The contract $(a^*(\cdot), b^*(\cdot), c^*(\cdot), \eta^*(\cdot))$ defined in the statement of Theorem 2 is optimal among all incentive feasible contracts in the continuous-time model.

Proof. Suppose that the contract is sub-optimal. Let $U(C)$ denote the expected utility to the entrepreneur from a contract C. There exists $\epsilon > 0$ and a feasible contract $(a(\cdot), b(\cdot), c(\cdot), \eta(\cdot))$ such that the promised payoff to the principal at date zero is $Q(0)$, but

$$U\left(a(\cdot), b(\cdot), c(\cdot), \eta(\cdot)\right) - U\left(a^*(\cdot), b^*(\cdot), c^*(\cdot), \eta^*(\cdot)\right) > \epsilon \tag{70}$$

Define the discrete-time approximations $(a_{\Delta t}(\cdot), b_{\Delta t}(\cdot), c_{\Delta t}(\cdot), \eta_{\Delta t}(\cdot))$ of the contract $(a(\cdot), b(\cdot), c(\cdot), \eta(\cdot))$ as in (64). There exists $\delta > 0$ such that

$$\left|U\left(a_{\Delta t}(\cdot), b_{\Delta t}(\cdot), c_{\Delta t}(\cdot), \eta_{\Delta t}(\cdot)\right) - U\left(a(\cdot), b(\cdot), c(\cdot), \eta(\cdot)\right)\right| < \epsilon/2 \tag{71}$$

for all $\Delta t < \delta$. By our earlier analysis of the discrete-time approximations,

$$U\left(a^*_{\Delta t}(\cdot), b^*_{\Delta t}(\cdot), c^*_{\Delta t}(\cdot), \eta^*_{\Delta t}(\cdot)\right) \geq U\left(a_{\Delta t}(\cdot), b_{\Delta t}(\cdot), c_{\Delta t}(\cdot), \eta_{\Delta t}(\cdot)\right). \tag{72}$$

It follows from (70), (71) and (72) that

$$\limsup_{\Delta t \to 0} U\left(a^*_{\Delta t}(\cdot), b^*_{\Delta t}(\cdot), c^*_{\Delta t}(\cdot), \eta^*_{\Delta t}(\cdot)\right) > U\left(a^*(\cdot), b^*(\cdot), c^*(\cdot), \eta^*(\cdot)\right),$$

which contradicts the result of Lemma 3. This completes the proof of the lemma. Q.E.D.

Proof of Theorem 3

By Propositions 1 and 2, $B_t^* \in (0, B_M)$ and $c(\cdot)$ is strictly concave in $(0, B_M)$. Hence, the denominator on the right hand side of (45) is negative. It follows that the sign of $\frac{dB_t^*}{dt}$ is determined by the sign of $c'(B_t^*)$. By Proposition 1, $c'(B_t^*) > 0$ for $B_t^* < 1$, $c'(B_t^*) = 0$ for $B_t^* = 1$, and $c'(B_t^*) < 0$ for $B_t^* > 1$. Hence, B_t^* is increasing whenever it is less than one, constant whenever it is equal to one and decreasing whenever it is greater than one.

The properties of B_t^* are, therefore, determined by those of B_T^*, which solves (46). Suppose that $\Omega_T < \lambda s^2$ or $\Omega_0 < \lambda(s^2 + \sigma_0^2 T)$ by (5). If $B_T^* > 1$ then $\Omega_T - \lambda s^2 B_T^* < 0$ and $c'(B_T^*) < 0$ by Proposition 1. It follows that B_T^* cannot satisfy (46). Hence, $B_T^* < 1$.

Suppose that $\Omega_T > \lambda s^2$. If $B_T^* \leq 1$ then $\Omega_T - \lambda s^2 B_T^* > 0$ and $c'(B_T^*) > 0$ by Proposition 1. It follows that B_T^* cannot satisfy (46). Hence, $B_T^* > 1$. Finally, if $\Omega_T = \lambda s^2$., it follows from the above arguments that $B_T^* = 1$ and, therefore, $B_t^* = 1$ for all t.

When $\Omega_T - \lambda s^2 < 0$, B_t^* are increasing and less than 1. Since σ_t is decreasing, by (47) b_t^* are increasing over time. When $\Omega_T - \lambda s^2 < 0$, B_t^* are decreasing and greater than 1. It follows from (47) that b_t^* decrease over time. Finally, when $\Omega_T = \lambda s^2$, $B_t^* = 1$ for all t so that $b_t^* = 1$ for all t by (47).

Depending on the sign of $\Omega_T - \lambda s^2$, the agent's total incentive intensities are either increasing and less than one, constant and equal to one, or decreasing and greater than one. By Proposition 1, the optimal investments are increasing when $B_t^* < 1$ and decreasing when $B_t^* > 1$. It follows that, when

B_t^* is either less than one and increasing or greater than one and decreasing, the optimal investment path is always increasing. If $B_t^* = 1$, the optimal investment path is constant.

By (33) and (36), the optimal effort is proportional to a positive power of $(B_t^*)^{\frac{1-\alpha}{\alpha}}(1 - \frac{\beta}{\gamma}B_t^*)$. Since $0 < B_t^* < B_M < \frac{\gamma}{\beta}(1-\alpha)$ by Proposition 2 and Assumption 2, the derivative of this expression with respect to B_t^* is positive. Thus, η_t^* is increasing (decreasing) when B_t^* is increasing (decreasing) and is constant when B_t^* is constant. ∎

Proof of Theorem 4

By (5), Ω_t increases with Ω_0 for each t. By (46) and the implicit function theorem

$$\frac{\partial B_T^*}{\partial \Omega_T} = \lambda s^2 - \frac{\gamma - \beta - \alpha\gamma}{\alpha\gamma} c^{''}(B_T^*) > 0, \tag{73}$$

where the last inequality follows from the fact that $B_T^* \in (0, B_M)$ by Proposition 2 and $c^{''}(B_T^*) < 0$ for $B_T^* \in (0, B_M)$ by Proposition 1. Hence, B_T^* increases with Ω_T and Ω_0.

Suppose that the assertion of the theorem regarding the effects of the initial degree of agent optimism on the path of total incentive intensities is false. There exist $\Omega_0^1 > \Omega_0^2$ and $u < T$ such that $B_u^*(\Omega_0^1) < B_u^*(\Omega_0^2)$, where the arguments signify the dependence of the total incentive intensities on the initial degree of agent optimism. Because $B_T^*(\Omega_0^1) > B_T^*(\Omega_0^2)$, it follows from the continuity of the total incentive intensity paths that there exists $t \in (u, T)$ such that $B_t^*(\Omega_0^1) = B_t^*(\Omega_0^2) = B$. Consider the ODE (45) over the interval $[t, T]$. Both $B_s^*(\Omega_0^1)$ and $B_s^*(\Omega_0^2)$ satisfy the ODE for $s \in [t, T]$ along with the initial condition $B_t^*(\Omega_0^1) = B_t^*(\Omega_0^2) = B$. By the uniqueness of the solution to the ODE and the initial condition, we must have $B_s^*(\Omega_0^1) = B_s^*(\Omega_0^2)$ for $s \in [t, T]$ so that, in particular, $B_T^*(\Omega_0^1) = B_T^*(\Omega_0^2)$, which contradicts the fact that $B_T^*(\Omega_0^1) > B_T^*(\Omega_0^2)$.

When $B_t^* < 1$ for $t \in [0, T]$, it follows from (33) and (36) that the investment and effort paths also increase with Ω_0. When $B_t^* > 1$ for $t \in [0, T]$, it follows from (33) that the investment at any date declines with the total incentive intensity so that the investment path decreases with Ω_0. Using the arguments at the end of the proof of Theorem 3, the effort at any date increases with the total incentive intensity so that the effort path increases with Ω_0.∎

Proof of Theorem 5

By (5), Ω_T decreases with the initial transient risk. It follows from (73) that

$$B_T^*(\sigma_1) \leq B_T^*(\sigma_2). \tag{74}$$

Case 1: $B_T^*(\sigma_1) \leq 1 \leq B_T^*(\sigma_2)$

The assertion that $B_t^*(\sigma_1) \leq 1 \leq B_t^*(\sigma_2)$ for all t follows immediately from Theorem 3. By the arguments in last part of the proof of Theorem 3, the agent's effort increases with his total incentive intensity so that $\eta_t^*(\sigma_1) < \eta_t^*(\sigma_2)$. By Proposition 1, the optimal investment function increases with the total incentive intensity when the latter is less than one and decreases when the latter is greater than one. It immediately follows that $c_t^*(\sigma_1)$ could be greater than, equal to, or less than $c_t^*(\sigma_2)$.
Case 2: $B_T^*(\sigma_1) \leq B_T^*(\sigma_2) \leq 1$

By (45),

$$\frac{dB_t^*}{dt}|_{t=T} = \frac{\frac{\sigma_0^2}{s^2+T\sigma_0^2}c^{'}(B_T^*)}{K\lambda s^2 - c''(B_T^*)},$$

where $K = \frac{\alpha\gamma}{\gamma-\beta-\alpha\gamma}$. Since $c^{'}(B_T^*) \geq 0$ when $B_T^* \leq 1$, the above implies that $\frac{dB_t^*}{dt}|_{t=T}$ increases with σ_0. It follows from (74) that there exists t such that $B_u^*(\sigma_1) < B_u^*(\sigma_2)$ for $u \in [t, T)$. Suppose that, contrary to the assertion of the theorem, there exists $t' < t$ such that $B_{t'}^*(\sigma_1) > B_{t'}^*(\sigma_2)$. By the continuity of the total incentive intensity paths, there exists $t" \in (t', t)$ such that $B_{t"}^*(\sigma_1) = B_{t"}^*(\sigma_2)$ and $B_u^*(\sigma_1) < B_u^*(\sigma_2)$ for $u \in [t", T)$.

Now note that

$$\frac{dB_t^*}{dt}|_{t=t"} = \frac{\frac{\sigma_0^2}{s^2+t"\sigma_0^2}c^{'}(B_{t"}^*)}{K\lambda s^2 - c''(B_{t"}^*)}.$$

The right hand side above is increasing in σ_0^2 (recall, $B < 1$ and therefore $c'(B) > 0$). It follows that there exists $\tilde{t} > t"$ such that $B_{\tilde{t}}^*(\sigma_1) > B_{\tilde{t}}^*(\sigma_2)$, which is a contradiction. Because $B_t^* < 1$, it follows from (33) and (36) that $\eta_t^*(\sigma_1) < \eta_t^*(\sigma_2)$ and $c_t^*(\sigma_1) < c_t^*(\sigma_2)$.

Case 3: $1 \leq B_T^*(\sigma_1) \leq B_T^*(\sigma_2)$

It suffices to show that the total incentive intensity paths corresponding to σ_1 and σ_2 cross at most once. Suppose that this is false. Since $B_T^*(\sigma_1) < B_T^*(\sigma_2)$, there exist $t_1 < t_2$ such that $B_{t_1}^*(\sigma_1) = B_{t_1}^*(\sigma_2)$, and $B_{t_2}^*(\sigma_1) = B_{t_2}^*(\sigma_2)$, and $B_u^*(\sigma_1) > B_u^*(\sigma_2)$ for $u \in (t_1, t_2)$. By (45),

$$\frac{dB_t^*(\sigma_1)|_{t=t_1}}{dt} = \frac{\frac{\sigma_1^2}{s^2+t_1\sigma_1^2}c^{'}(B_{t_1}^*(\sigma_1))}{K\lambda s^2 - c''(B_{t_1}^*(\sigma_1))} < \frac{\frac{\sigma_2^2}{s^2+t_1\sigma_2^2}c^{'}(B_{t_1}^*(\sigma_2))}{K\lambda s^2 - c''(B_{t_1}^*(\sigma_2))} = \frac{dB_t^*(\sigma_2)|_{t=t_1}}{dt},$$

where the inequality follows because $\sigma_1 > \sigma_2$, $B_{t_1}^*(\sigma_1) = B_{t_1}^*(\sigma_2)$ and $c^{'}(B_{t_1}^*(\sigma_1)) < 0$. It follows from the above that there exists $t_1^{'} > t_1$ such that $B_u^*(\sigma_1) < B_u^*(\sigma_2)$ for $u \in (t_1, t_1^{'})$, which contradicts the fact that $B_u^*(\sigma_1) > B_u^*(\sigma_2)$ for $u \in (t_1, t_2)$.

The results regarding the investments and effort follows from the fact that investment decreases with the total incentive intensity when the latter is greater than one, while the effort increases. ∎

Proof of Theorem 6

By (5), Ω_T increases with s. It follows from (46) and the implicit function theorem that B_T^* could increase or decrease with s.

Case 1: $B_T^*(s_1) < B_T^*(s_2)$.

a) Suppose $B_T^*(s_1) \leq 1 \leq B_T^*(s_2)$. The assertions in the statement of the theorem follow using arguments identical to those uses to establish Case 1 in the proof of Theorem 5.

b) Suppose $B_T^*(s_1) < B_T^*(s_2) \leq 1$. It suffices to show that the total incentive intensity paths corresponding to s_1 and s_2 cross at most once. Suppose this is false. There exist $t_1 < t_2$ such that $B_{t_1}^*(s_1) = B_{t_1}^*(s_2)$, and $B_{t_2}^*(s_1) = B_{t_2}^*(s_2)$, and $B_u^*(s_1) > B_u^*(s_2)$ for $u \in (t_1, t_2)$. By (45),

$$\frac{dB_t^*(s_1)|_{t=t_1}}{dt} = \frac{\frac{\sigma_0^2}{s_1^2+t_1\sigma_0^2}c^{'}(B_{t_1}^*(s_1))}{K\lambda s_1^2 - c''(B_{t_1}^*(s_1))} < \frac{\frac{\sigma_0^2}{s_2^2+t_1\sigma_0^2}c^{'}(B_{t_1}^*(s_2))}{K\lambda s_2^2 - c''(B_{t_1}^*(s_2))} = \frac{dB_t^*(s_2)|_{t=t_1}}{dt},$$

where the inequality follow because $s_1 > s_2$, and $B_{t_1}^*(s_1) = B_{t_1}^*(s_2)$. It follows from the above that

there exists $t_1^{'} > t_1$ such that $B_u^*(s_1) < B_u^*(s_2)$ for $u \in (t_1, t_1^{'})$, which contradicts the fact that $B_u^*(s_1) > B_u^*(s_2)$ for $u \in (t_1, t_2)$.
c) Suppose $1 \leq B_T^*(s_1) < B_T^*(s_2)$. If the assertion of the theorem is false, there exists $t_1 < T$ such that $B_{t_1}^*(s_1) = B_{t_1}^*(s_2)$ and $B_u^*(s_1) < B_u^*(s_2)$ for $u \in (t_1, T]$. By (45),

$$\frac{dB_t^*(s_1)|_{t=t_1}}{dt} = \frac{\frac{\sigma_0^2}{s_1^2+t_1\sigma_0^2} c^{'}(B_{t_1}^*(s_1))}{K\lambda s_1^2 - c''(B_{t_1}^*(s_1))} > \frac{\frac{\sigma_0^2}{s_2^2+t_1\sigma_0^2} c^{'}(B_{t_1}^*(s_2))}{K\lambda s_2^2 - c''(B_{t_1}^*(s_2))} = \frac{dB_t^*(s_2)|_{t=t_1}}{dt},$$

where the inequality follows because $s_1 > s_2$ and $c^{'}(B_{t_1}^*(s_1)) < 0$ as $B_{t_1}^*(s_1) > 1$. Hence, there exists $t_1^{'} > t_1$ such that $B_u^*(s_1) > B_u^*(s_2)$ for $u \in (t_1, t_1^{'})$, which is a contradiction. The results describing the investment and effort paths follow using arguments that are very similar to those used in the proofs of the previous theorems and are omitted for brevity.
Case 2: $B_T^*(s_1) > B_T^*(s_2)$.
a) Suppose $B_T^*(s_1) \geq 1 \geq B_T^*(s_2)$. The assertions in the statement of the theorem follow using arguments identical to those used to establish Case 1 in the proof of Theorem 5.
b) Suppose $1 \geq B_T^*(s_1) > B_T^*(s_2)$. If the assertion of the theorem is false, there exists $t_1 < T$ such that $B_{t_1}^*(s_1) = B_{t_1}^*(s_2)$ and $B_u^*(s_1) > B_u^*(s_2)$ for $u \in (t_1, T]$. By (45),

$$\frac{dB_t^*(s_1)|_{t=t_1}}{dt} = \frac{\frac{\sigma_0^2}{s_1^2+t_1\sigma_0^2} c^{'}(B_{t_1}^*(s_1))}{K\lambda s_1^2 - c''(B_{t_1}^*(s_1))} < \frac{\frac{\sigma_0^2}{s_2^2+t_1\sigma_0^2} c^{'}(B_{t_1}^*(s_2))}{K\lambda s_2^2 - c''(B_{t_1}^*(s_2))} = \frac{dB_t^*(s_2)|_{t=t_1}}{dt},$$

where the inequality follows because $s_1 > s_2$ and $c^{'}(B_{t_1}^*(s_1)) > 0$ as $B_{t_1}^*(s_1) < 1$. Hence, there exists $t_1^{'} > t_1$ such that $B_u^*(s_1) < B_u^*(s_2)$ for $u \in (t_1, t_1^{'})$, which is a contradiction.
c) Suppose $1 \leq B_T^*(s_2) < B_T^*(s_1)$. It suffices to show that the total incentive intensity paths corresponding to s_1 and s_2 cross at most once. Suppose that this is false. There exist $t_1 < t_2$ such that $B_{t_1}^*(s_1) = B_{t_1}^*(s_2)$, $B_{t_2}^*(s_1) = B_{t_2}^*(s_2)$, and $B_u^*(s_1) < B_u^*(s_2)$ for $u \in (t_1, t_2)$. By (45),

$$\frac{dB_t^*(s_1)|_{t=t_1}}{dt} = \frac{\frac{\sigma_0^2}{s_1^2+t_1\sigma_0^2} c^{'}(B_{t_1}^*(s_1))}{K\lambda s_1^2 - c''(B_{t_1}^*(s_1))} > \frac{\frac{\sigma_0^2}{s_2^2+t_1\sigma_0^2} c^{'}(B_{t_1}^*(s_2))}{K\lambda s_2^2 - c''(B_{t_1}^*(s_2))} = \frac{dB_t^*(s_2)|_{t=t_1}}{dt},$$

where the inequality follow because $s_1 > s_2$, and $B_{t_1}^*(s_1) = B_{t_1}^*(s_2)$ and $c^{'}(B_{t_1}^*(s_1)) < 0$. Hence, there exists $t_1^{'} > t_1$ such that $B_u^*(s_1) > B_u^*(s_2)$ for $u \in (t_1, t_1^{'})$, which contradicts the fact that $B_u^*(s_1) < B_u^*(s_2)$ for $u \in (t_1, t_2)$. The results describing the investment and effort paths follow using arguments that are very similar to those used in the proofs of the previous theorems.

Proof of Proposition 5

By (33) and (36), the discretionary output in any period is proportional to a positive power of $B_t^*(1 - \frac{\beta}{\gamma} B_t^*)^{\alpha\gamma/\beta}$. The derivative of this expression is positive if and only if $B_t^* < \frac{\gamma}{\beta+\alpha\gamma}$. Finally, by Assumption 1, $1 < \frac{\gamma}{2\beta+\gamma\alpha} < \frac{\gamma}{\beta+\gamma\alpha}$. ∎

Proof of Theorem 7

By (49), and the fact that $E_0^{Pr}[\Theta] = \mu_0^{Pr}$,

$$H_t = \mu_0^{Pr} T + \int_0^T \Phi(c_t^*, \eta_t^*) dt \tag{75}$$

By Proposition 5, the discretionary output $\Phi(c_t^*, \eta_t^*)$ increases with B_t^* if $B_t^* < 1$. It follows from (75), therefore, that H_0 increases with the total incentive intensity path.

Suppose the agent is exuberant and $B_0^* < \frac{\gamma}{\beta+\alpha\gamma}$. By Theorem 3, the agent's total incentive intensities decline over time so that $B_t^* < \frac{\gamma}{\beta+\alpha\gamma}$ for $t \in [0, T]$. By Proposition 5, the discretionary output $\Phi(c_t^*, \eta_t^*)$ increases with B_t^* if $B_t^* < \frac{\gamma}{\beta+\alpha\gamma}$. By (75), H_0 increases with the total incentive intensity path. If $B_0^* > \frac{\gamma}{\beta+\alpha\gamma}$, then by Theorem 3, there exists $t^* \in [0, T]$ such that $B_t^* > \frac{\gamma}{\beta+\alpha\gamma}$ for $t < t^*$ and $B_t^* < \frac{\gamma}{\beta+\alpha\gamma}$ for $t > t^*$. By Proposition 5, the discretionary output $\Phi(c_t^*, \eta_t^*)$, therefore, increases with B_t^* if $t < t^*$, and decreases with B_t^* if $t > t^*$ It follows from (75), therefore, that H_0 could either increase or decrease with the total incentive intensity path. ∎

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