Delegated Learning in Asset Management*

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ABSTRACT

We develop a tractable framework of delegated asset management with information acquisition in a multi-asset economy in which fund managers face moral hazard in portfolio allocation decisions. In our setting, benchmarking arises endogenously as part of the contract between investors and active fund managers, and we highlight a novel contract externality in which prices feed back to the contract. Our framework sheds light on a tension between the incentives of active managers to trade on their acquired private information, and both their hedging demand for the benchmark portfolio and the risk-return trade-off of the underlying assets. These insights allow us to uncover a potential gap between our model-implied measure and several widely-adopted empirical statistics intended to capture managerial ability. In a multi-period extension of our model, we propose a new measure of fund manager skill.

Keywords: Fund Manager Compensation, Endogenous Benchmark, Moral Hazard, Information Acquisition


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1 Introduction

There is growing concern that active fund managers lack the superior ability in garnering higher returns to justify their higher fees compared to their passive counterparts. The existing literature has focused on either improving empirical measures to evaluate the unobservable skill of fund managers, or on developing theories to justify the lack of empirical support for their superior ability. Despite the progress of this fast growing literature, the relationship between fund manager ability and the incentives that they face, in equilibrium, is still not well-understood. In this paper, we ask to what extent such unobservable ability is an outcome of the incentives provided to active managers in equilibrium, and aim to shed light on how to measure skill in the asset management industry.

To investigate this issue, we cast the information acquisition and portfolio allocation decisions of a delegated asset manager as a principal-agent problem between the active fund manager and its investors. We refer to this as the delegated learning channel. We study an economy in which asset managers can trade on behalf of investors in a multi-asset financial market, similar to that in Admati (1985). In the model, a fraction of fund managers are active managers, who are able to exert costly effort to learn about the aggregate and asset-specific components of the payoffs of the assets (Kacperczyk et al. (2016)), while the rest of managers are passive managers who form portfolios after observing the prices. Investors delegate their asset allocation decisions to both active and passive funds. The inability of investors to observe the effort and portfolio decisions of the active managers, however, forces investors to offer a contract that is incentive compatible to active fund managers. In

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1 Consistent with this view, in recent years there has been an accelerating shift in fund flows from active to passive strategies. Since 2005, actively managed equity and fixed income funds have lost fund flows to passive strategies globally. According to MorningStar, over last decade, actively run U.S. stock funds saw net outflows every year, totaling about $600 billion, while their indexed counterparts saw net inflows of approximately $700 billion. See http://www.marquetteassociates.com/research/a-continued-shift-from-active-to-passive-in-u-s-equities.

2 The existing literature has developed several theories to help explain the lack of empirical support that active managers have superior ability, including that fund performance exhibits decreasing-returns-to-scale (Berk and Green, 2004), that managers choose investments based on their benchmark and flow-performance sensitivity (Brennan (1993), Admatı and Pfleiderer (1997), Buffa et al. (2014)), and that skill reflects a choice to acquire information over the business cycle (Kacperczyk et al. (2014), Kacperczyk et al. (2016)).
equilibrium, these incentives feed into asset prices as active managers trade on their private information in financial markets, which then feed back into the determinants of the contract in the principal-agent problem between the manager and investors.

The affine contract for active fund managers in our setting features three components: a fixed fee, a performance-based reward that evaluates a fund manager for its performance, and benchmarking relative to the ex ante mean-variance efficient portfolio. In contrast to such frameworks as those of Basak and Pavlova (2013) and Buffa et al. (2014), the optimal benchmarking endogenously arises in our model, and it represents the outside option to investors at the time of contracting.

Given that investor and active managers are asymmetrically exposed to noise in managers private information, the optimal choice of performance-based incentive deviates from perfect risk-sharing. This departure introduces a role for benchmarking to help align preferences which is absent in Admati and Pfleiderer (1997), and has impact on effort choices and portfolio decisions of active managers in equilibrium. First, by benchmarking, the investor effectively endows the active manager with a tilted long position in the benchmark portfolio, which leads it to hedge its benchmark risk with passive managers. This tilt impacts the level of risk-sharing between the investor and the active manager. Second, since the performance-based reward influences the aggressiveness with which active fund managers trade on their private information, it feeds into the informativeness of asset prices in equilibrium. This introduces a novel contract externality, through which the performance-based piece impacts the uncertainty that active fund managers face at the industry level when choosing their portfolio, and consequently their effort to acquire private information.

Our model has novel implications for identifying skill among fund managers. We highlight a gap between our model-implied measure of fund manager skill and empirical statistics meant to capture asset management ability, such as the active share proposed by Cremers and Petajisto (2009) and the return gap of Kacperczyk et al. (2008). The implication on performance evaluation is driven by the tension between the incentivized active position
taken by active managers based on their private information, and both their hedging demand for the benchmark portfolio and the risk-return trade-off of the underlying assets. We analyze two sets of comparative statics to illustrate the intuition behind these tensions.

When the overall level of payoff uncertainty decreases, the representation of risky assets in the benchmark portfolio increases because the benchmark is determined by the ex ante risk-return tradeoff. By benchmarking, the active fund managers are endowed with a tilted long position in the benchmark portfolio, which leads it to hedge its benchmark risk by taking a short position in it. The hedging demand hence drives down the equilibrium prices of assets in the benchmark portfolio. At the same time, there is less benefit for active managers in acquiring information as uncertainty decreases, and active managers exerts less effort. As a result, active managers trade less aggressively on their private information, reducing their active position as their holdings become dominated by the benchmark portfolio. As the holdings of active managers approach that of the benchmark, consequently, the skill level is correctly reflected in our theoretical analogues of the active share and the return gap.

When asset payoffs become more correlated, however, our analysis cautions in the interpretation of these empirical proxies for manager skills. Increasing the correlation between asset payoffs lowers the effort exerted by active managers to learn. In contrast, these managers may appear more active because their benchmark takes smaller positions in the risky assets because of the diminished benefits from diversification, and because the expected excess return of the underlying assets increases. The gap between the empirical proxies and their theoretical analogues in our paper highlights the importance of endogenizing the benchmark for theoretical predictions.

We then propose an incentive-based measure of manager skills by investigating a dynamic extension of our baseline model in which trading by managers occurs over multiple periods. This dynamic extension illustrates that having multiple periods introduces intertemporal incentives for fund managers to acquire private information and, more importantly, can provide investors with a time-series of past fund behavior to improve monitoring. We show that
the historical variance of a fund’s return gap, downweighted by the dispersion of asset payoffs, provides a consistent measure of average portfolio selection skill. Learning by investors through this channel can help explain the nonlinear relationship between performance and fund flows observed empirically.

**Literature Review**  Our work is related to the literature on delegated asset management under asymmetric information. García and Vanden (2009) and Gárleanu and Pedersen (2015) explore the implications for market efficiency of the formation of mutual funds in the presence of costly information acquisition in a single asset setting. García and Vanden (2009) also consider a model of delegated asset management with information acquisition, yet their focus is on market efficiency, and they assume that managers sell funds to households and pay a fixed fee to become informed in the spirit of Grossman and Stiglitz (1980). Our work focuses on an affine contract between investors and fund managers in a multi-asset principal-agent setting. Leippold and Rohner (2012) also examine delegated asset management with benchmarking and information acquisition in an equilibrium framework, yet their focus is on asset pricing implications with an exogenous benchmark in the absence of an agency conflict between investors and managers. Kapur and Timmermann (2005) investigate the impact of relative performance contracts on the equity premium and on portfolio herding. Dybvig et al. (2010) and He and Xiong (2013) consider the market-timing benefits of benchmarking in a partial equilibrium setting. Kyle et al. (2011) investigates the incentives to acquire information under delegated asset management for a large informed fund, in the spirit of Kyle (1985), while Glode (2011) and Savov (2014) microfound delegated asset management as a vehicle for investors to hedge their background risk. Huang (2015) studies the market for information brokers in an equilibrium setting with optimal contracting to explain home bias, comovement in asset idiosyncratic volatility, and the possibility of herding and equilibria multiplicity.

This paper is connected to the growing literature on equilibrium asset pricing with flexible
information acquisition. Van Nieuwerburgh and Veldkamp (2009, 2010) and Kacperczyk et al. (2016) study the flexible information acquisition problem faced by investors who have limited attention that they can allocate to learning about risky asset payoffs, the latter of which focuses on business cycle implications. Maćkowiak and Wiederholt (2012) investigate the information acquisition decisions of investors who have limited liability, while Huang et al. (2016) models information acquisition as part of a dynamic reputation game between the fund and its investors. In contrast to these studies, we model the information acquisition of managers as being subject to agency issues within an equilibrium framework.

In addition, our work is also related to the literature on manager incentives and benchmarking in the asset management industry. Basak and Pavlova (2013) and Buffa et al. (2014) investigate the asset pricing implications of benchmarking against an exogenous index in a multi-asset setting, with Buffa et al. (2014) embedding benchmarking in a principal-agent framework. Buffa and Hodor (2017) explore the asset pricing implications of heterogeneous benchmarking. Starks (1987) studies the role of symmetric versus bonus performance-based contracts in incentivizing asset managers. Brennan (1993) examines the CAPM implications of delegated management with both exogenous and optimal benchmarking. Admati and Pfleiderer (1997) analyzes benchmarking and manager incentives in a partial equilibrium framework in which managers have superior information to investors, while Ou-Yang (2003) investigates the optimal affine contract in a finite horizon setting in which managers bear a time-varying cost of investing in a portfolio. van Binsbergen et al. (2008) explores how benchmarking can overcome moral hazard issues that arise with decentralization. Cuoco and Kaniel (2011) study the implications for asset pricing when manager compensation is linked to a benchmark, and Li and Tiwari (2009) study nonlinear performance-based contracts in the presence of benchmarking. In our work, we derive the optimal benchmark jointly with the optimal affine contract and equilibrium prices, and study their empirical implications for intermediary holdings and asset returns.
2 A Model of Delegated Asset Management

In this section, we present a model of delegated asset management with flexible information acquisition in a multi-asset principal-agent framework. In this economy, there are investors who can allocate their wealth between an active fund, which is subject to agency issues, and a passive fund. We first introduce the asset environment, and then discuss the problem faced by each type of agent. Finally, we define the asset market equilibrium.

2.1 The Environment

Asset Fundamentals There are three dates \( t = \{0, 1, 2\} \). There are \( N \) assets with risky payoffs \( f_i, i \in \{1, 2, \ldots, N\} \), which realize at date 2 that satisfy the following decomposition:

\[
\begin{cases}
  b_1 \theta_1 \\
  a_i \theta_i + b_i \theta_1, \ i \in \{2, \ldots, N\}
\end{cases}
\]

The common component \( \theta_1 \) can be viewed as aggregate payoff risk, with \( b_i \) being the loading on this aggregate payoff risk of the asset, while the \( a_i \theta_i, i \in \{2, \ldots, N\} \) are the asset-specific components of the risky asset payoffs. This payoff structure we employ is similar to that in Buffa et al. (2014) and Kacperczyk et al. (2016). For interpretation of \( \theta_1 \) as aggregate payoff risk, we assume that \( a_1 = 0, b_1 = 1 \) and that the first asset is a composite asset of the remaining assets in the economy with a payoff that loads only on this aggregate payoff risk.\(^3\) Asset \( i \) has price \( P_i \) at \( t = 1 \), and we stack the \( N \) prices into the \( N \times 1 \) vector \( P \). In what follows, bold symbols represent vectors. For convenience, we define the vector \( \Theta = \left[ \theta_1 \theta_2 \cdots \theta_N \right]' \) such that:

\[
f = F \Theta,
\]

\(^3\)Kacperczyk et al. (2016) employ a similar assumption for the asset payoff structure. While not essential for our analysis, it helps with exposition by ensuring that the map from risk factors \( \{\theta_1, \{\theta_i\}_{i \in \{2, \ldots, N\}}\} \) to asset payoffs \( \{f_i\}_{i \in \{1, \ldots, N\}} \) is invertible.
for the $N \times N$ matrix $F$, which is invertible since $F$ is lower triangular provided that $b_i > 0 \ \forall i$. In our setting, aggregate risk arises through the correlation structure of asset payoffs, and is represented by the common fundamental $\theta_i$.\footnote{This is in contrast to Kacperczyk et al. (2016), where aggregate risk takes the form of the asset fundamental with a higher supply variance. Our derivations will, in fact, be valid more generally for any arbitrary invertible matrix $F$.} In addition to the $N$ assets, there is a risk-free asset, which can be viewed as asset 0, in perfectly elastic supply with gross return $R_f > 1$.

We assume that all agents in our model have a normal prior over $\Theta$, and initially believe that $\Theta \sim N(\bar{\Theta}, \tau^{-1}I_{d_N})$, where $\tau$ is the common precision of the prior over the hidden factors driving asset payoffs. One can view the prior as reflecting all publicly available information about the asset payoffs, such as financial disclosures, earnings announcement, and macroeconomic news that agents have before contracting at date 0.

**Agents** There is a unit continuum of investors, each with initial wealth $W_0$, that can allocate this wealth at date 0 between active and passive management. Investor $i$ invests a fraction $y_i$ of its wealth in an active advisory fund, and the remaining $1 - y_i$ in the passive advisory firm. This assumption reflects that, in practice, investors can freely allocate capital across investment opportunities, and both active and passive vehicles represent sizable fractions of the asset management industry. Although the funds in which investors delegate their wealth can borrow and lend freely at the risk-free rate, we assume for simplicity that investors cannot. As a consequence, $y_i \in [0, 1]$. The fund allocation decisions of all investors are publicly observable.

There is an active and a passive advisory firm. The active firm manages a mass $\chi \in [0, 1]$ of active funds and the passive firm managers a mass $1 - \chi$ of passive funds, and each fund employs one manager. Managers have no initial wealth. As the number of active funds is limited, active management is in short supply for investors, and this mass $\chi$ is known to all agents. We defer discussion of how investors allocate their wealth to these funds to subsection 2.5.
While the passive firm employs managers to implement a passive strategy that is known to investors, the active firm pays managers to take active positions by exerting costly effort to acquire private information. Their information acquisition activities and portfolio allocation decisions, however, are unobservable to both investors and the active advisory firm, and this gives rise to an agency conflict between investors and active fund managers. This is what we refer to as the delegated learning channel.

The active advisory firm offers one compensation contract that is incentive compatible to all managers of its funds that maximizes investor utility subject to their participation. To focus on the agency conflict between investors and active managers, we abstract from issues of decentralization, as in van Binsbergen et al. (2008), and the tournament incentives featured in, for instance, Kapur and Timmermann (2005), which can lead to herding when manager participation constraints bind. Our focus on manager compensation incentives distinguishes our setting from that in García and Vanden  (2009) and Gáreleanu and Pedersen (2015), who investigate how asset management companies set fees for investors, which are relatively stable in practice.

Since our model features a static setting, we are unable to address implicit incentives that arise from career concerns and fund flows. In addition, as there is only one observation of a fund manager’s performance, there is little that can be inferred about the manager’s skill in acquiring information, as noted in Admati and Pfleiderer (1997). We return to these issues when we consider the dynamic extension of the model.

We now discuss the problem faced by each of these agents, in turn.

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5Brown and Davies (2016) also studied the moral hazard in the active asset management industry in a partial equilibrium framework. They assume the effort exerted by managers are directly linked to returns, while we focus on the incentives to acquire costly information.

6An earlier version allowed the compensation contract to condition on the realized performance of the passive fund. This mainly modified the optimal benchmark. Since it did not add much additional insight for the additional notational complexity, we omit it to simplify the exposition.

7Ibert et al. (2017) recently provided evidence on the importance of fund size as part of the manager compensation.
2.2 Passive Fund

Since it is not the focus of our analysis, we keep our specification of the passive fund as simple as possible. The passive advisory firm employs managers who at date 1 allocate all their capital to holding the mean-variance efficient portfolio, $\omega^D_1$, after observing market prices, $P$, and we assume passive managers trade with the same coefficient of absolute risk aversion, $\gamma > 0$. \footnote{This simplification avoids the issue that passive managers would want to hedge investors’ background exposure to active funds if they instead maximized investors’ utility. The mean-variance efficient portfolio is also a portfolio of independent interest, since it is the portfolio investors would choose if they invested directly in financial markets at date 1, and the optimal benchmark in Dybvig et al. (2010).}

Each passive fund has a final AUM at date 2, $W_2^D$ given by:

$$W_2^D = R^f W_0 + \omega^D_1 (f - R^f P).$$

Since the mean-variance efficient portfolio is publicly observable, there is no agency conflict between passive managers and investors, and they will charge a fixed fee for their services. \footnote{Since there is no agency conflict, investors can extract the surplus from the relationship to make the participation constraint of the passive fund manager bind. If the passive fund manager is risk-averse, then the cheapest form of compensation is a fixed fee, which we normalize to zero.} For parsimony, we normalize this fee to zero.

If the unit mass of investors, on aggregate, allocate a fraction $1 - y_i$ of their capital $W_0$ to the passive advisory firm, then they will receive an aggregate dollar payoff $(1 - y_i)W_2^D$ that can be distributed among them.

2.3 Active Fund

We assume that investors are randomly allocated to a mass $\chi$ of fund managers by the active advisory firm, and are therefore exposed to fund-specific risk. There is a many-to-one matching protocol because there is a unit mass of investors who are matched with a smaller set $\chi$ of fund managers. Though investors could achieve full diversification by investing in all active funds, this is, in part, an artifact of the information structure, and also at variance
Fund managers in the active fund each face a portfolio choice problem at date 1. Given their information and initial AUM $W_0$, active managers choose a portfolio allocation strategy $\omega^S_1$ at date 1 across the $N$ assets after observing market prices $P$ and their private information, so that the final AUM $W^S_2$ is given by:

$$W^S_2 = R^f W_0 + \omega^S_1 (f - R^f P).$$

As described above, in addition to a portfolio choice problem, active managers must exert costly effort at date 0 to acquire their private information about asset payoffs at date 1. While asset prices are publicly observable, active managers also acquire a vector of noisy private signals $s_j$ about $\theta_1$ and the asset-specific component of asset payoffs $\theta_i$, $i \in \{2, ..., N\}$. They can exert a vector of efforts $e = e^t_1 N \times 1 \geq 0$, with $e \geq 0$ element-by-element, to reduce the variance of these signals $\Sigma (e)$.

Specifically, active manager $j$ receives a vector of noisy signals $s_j$ about $\Theta$ given the effort level $e_j$:

$$s_j = \Theta + \Sigma_j (e_j)^{1/2} \varepsilon_j,$$

where $\varepsilon_j \sim N(0_{N \times 1}, I_{N})$ is independent across $j$ and satisfies the Strong LLN $\int_{-\infty}^{\infty} \varepsilon_j d\Phi (\varepsilon_j) = 0_{N \times 1}$ for $\Phi (\cdot)$, the CDF of the standard normal distribution. Following Kacperczyk et al. (2014), we assume that $\Sigma_j(e_j)$ is a diagonal matrix with entry $K^{-1}_{ii}(e_{ij})$ that satisfies a monotonicity condition. We assume that $\Sigma_j(e_j)$ is diagonal so that there is a direct link between the effort manager $j$ exerts to learn about the $i^{th}$ component of $\Theta$, $e_{ij}$, and the precision of the signal manager $j$ receives about that component, $s_{ij}$. The monotonicity condition

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10Empirically, investor fund flows are correlated with past performance, which we would not expect to observe if investors diversified their holdings across active funds. Furthermore, the premise of active funds is to be exposed to “active alpha” strategies, and much of the literature, in the spirit of Berk and Green (2004), has focused on the problem of investors identifying managers that can deliver superior performance. García and Vanden (2009) make a similar assumption in that, despite holding a portfolio of funds, households are still exposed to fund idiosyncratic risk.

11The monotonicity condition we require is that $\Sigma_j(e_j''') - \Sigma_j(e_j'')$ is positive-semi definite (PSD) whenever $e_j' \geq e_j''$.
we impose ensures that a higher level of effort (weakly) implies the manager receives more informative signals. To ensure prices are always informative, we regulate \( \Sigma (e_j) \) by assuming that \( \sup_i \Sigma (0_{N \times 1}) \leq M^{-1} < \infty \). In what follows, we parameterize \( K_i (e_{ij}) = M + e_{ij} \).

One can view this observation of private information by a fund manager as their security selection or “stock picking ability”.

Active fund managers have CARA preferences over their compensation from investors \( C_0^S \) and the monetary cost of exerting effort to acquire private information:

\[
u (C_0^S ; \omega^S_1, e) = - \exp (H(e) - \gamma_M C_0^S),\]

where \( \gamma_M \) is the coefficient of absolute risk aversion and \( H (\cdot) \) is the dollar cost for effort \( e \), an increasing and (strictly) convex function in each of its arguments, such that \( \partial_i H (e) > 0 \) and \( \partial_{ii} H (e) \geq 0 \), and \( H (0_{N \times 1}) = 0 \) as a normalization. We specialize \( H (e) \) to the case that \( H (e) = \frac{1}{2} h (e' 1_{N \times 1}) \), where \( h' (\cdot) > 0, h'' (\cdot) \geq 0, \) and \( h(0) = 0 \). This functional form induces potential substitutability in manager learning decisions, and therefore a tradeoff to learning too much about one source of asset-specific risk.

An agency conflict arises because the effort that active managers exert, and their portfolio choice, are not observable. An active manager must therefore find it optimal to follow the recommendation of investors, which gives rise to the incentive compatibility (IC) constraint:

\[
e \in \arg \sup_{e \in \mathbb{R}^N} E \left[ \sup_{\omega \in \mathbb{R}^N} E \left[ u (C_0^S ; \omega^S_1, e') \mid F_j \right] \right] \quad (IC),
\]

where \( F_j \) is the fund manager’s information set, and the optimization implies a natural timing to their decisions. The active manager first determines the effort to exert based on the compensation contract \( C_0^S \) with investors at date 0. At date 1, the active manager observes prices and private signals, and makes portfolio allocation choice. The active fund manager’s information set is then the sigma algebra generated from observing the vector of

\footnote{Our results will be valid in the limit that \( M \searrow 0 \).}
prices $\mathbf{P}$ and its private signals $s_j$, $\mathcal{F}_j = \sigma(\mathbf{P}, s_j(e_j))$.

In addition to the IC constraint, active fund managers are also subject to a participation constraint:

$$E\left[u\left(C_0^S, \omega_1^S, \mathbf{e}\right) \mid \mathcal{F}_j\right] \geq u_0,$$

where $u_0$ is the reservation utility faced by the active fund managers.

In what follows, the IC constraint will always bind, and we primarily will consider parameter restrictions to focus on the case in which the participation constraint does not bind. Since investors will have to be indifferent to investing the marginal dollar for the market for intermediaries to clear, they implicitly face a participation constraint that will always bind when splitting the surplus of active management with the active fund managers. If the participation constraint also binds for managers, then this imposes a limit on the effort that active managers will exert to acquire private information.

We assume that the signal noise of all active fund managers is uncorrelated, so that a Weak Law of Large Numbers, in the spirit of Uhlig (1996), holds. Once returns are realized, the aggregate return of the active fund is then independent of the fund-specific risk of any individual active fund. We assume a random matching protocol between investors and active fund managers, so that the advisory firm then pays out to each investor this aggregate return adjusted by the individual performance of the fund manager with whom the investor has been matched. Due to this matching protocol, wealth is neither created nor destroyed.

Finally, we assume the active advisory firm chooses the compensation contract, $C_0^S$, that it offers to all fund managers, to maximize the return to investors such that the market for intermediaries clears.\footnote{With this protocol, it is not important whether the advisory firm, the active fund managers, or the investors offer the contract. Maximizing the surplus of active management to investors allows managers to extract the most rent from the relationship, as their fixed fee will capture the marginal benefit of a dollar allocated to active instead of passive funds.}
2.4 Investors

Investors have CARA preferences over the final AUM at date 2, $W^S_2$. They choose their asset allocation policy $y_i$ across the active and passive fund of funds to maximize their utility subject to incurring the cost of compensating the active fund manager:

$$U (W^I_2, y_i; C^S_0) = -\exp \left( -\gamma \left( y_i (W^S_2 - C^S_0) + (1 - y_i)W^D_2 \right) \right),$$

where $\gamma > 0$ is their coefficient of absolute risk aversion.

The investors solve the optimization problem when allocating their capital:

$$V_0 = \sup_{y_i} E^{e(C^S_0)} [U (W^I_2, y_i; C^S_0)],$$

subject to the PC and IC constraints, where $E^{e(\cdot)} [\cdot]$ is understood as the expectation under the probability distribution induced by the recommended effort level $e(C^S_0)$. Consequently, $V_0$ is the indirect utility of investors.

Similar to Ou-Yang (2003), Kapur and Timmermann (2005), Buffa et al. (2014) and Sotes-Paladino and Zapatero (2017), we restrict our attention to the space of affine contracts between investors and active fund managers through the active advisory firm for several reasons. First, is to advance our understanding of how such incentive schemes extend to an equilibrium setting. Stoughton (1993) and Admati and Pfleiderer (1997), for instance, both show in partial equilibrium that affine contracts fail to provide incentives for managers to acquire the private information with CARA preferences. We analyze affine contracts in this linear paradigm and show that affine contracts can indirectly provide incentives to active managers when asset prices aggregate their private information. This also gives rise to a contract externality in that the contract offered to each active manager does not take into account that they affect incentives of other managers through their impacts on asset prices. Second, option-like incentives hamper risk-sharing between investors and managers, and they
are likely to worsen the effort that managers exert to acquire private information because the down-side protection gives them incentives to take risk by remaining less informed.\footnote{Starks (1987) shows that linear contracts lead to an optimal portfolio risk exposure by managers, but an under-provision of effort compared to the first-best, while asymmetric contracts with bonus incentives lead to both a suboptimal risk exposure and an even lower level of effort than the symmetric case. Sotes-Paladino and Zapatero (2017) also finds that option-like payoffs can be suboptimal to linear contracts with asymmetric information between investors and managers, though in the absence of moral hazard.} Finally, since we are solving for noisy rational expectations within the linear paradigm of Grossman and Stiglitz (1980) and Hellwig (1980), such a restriction may be seen as a natural extension of the focus on linear equilibria.

We therefore focus on the set of linear contracts \( C_0^S \) of the form:

\[
C_0^S = \rho_0 + \rho_S (W_2^S - R^f W_0) + \rho'_R (f - R^f P).
\]

The compensation schedule \( C_0^S \) is contingent on outcomes that are observable to investors at date 2, the realized portfolio return per share of the fund \( W_2^S - R^f W_0 \) and the realized excess payoffs of the risky assets \( f - R^f P \).\footnote{We also considered a version in which the compensation contract conditions on the realized return of passive funds \( W_2^D \). Since \( W_2^D \) is based on public information, and is exogenous to the choices of any active fund manager, it has no substantive impact on information acquisition. It does, however, affect the hedging incentives in their portfolio choice. See Kapur and Timmermann (2005) for this type of incentive contract.} Conditioning compensation on the realized excess payoff of the portfolio potentially helps to align the incentives of the active manager and investor by giving the manager an equity stake in the portfolio. In addition, allowing the compensation schedule to vary with observed excess payoffs \( f - R^f P \) can also improve incentives by providing flexibility for the contract to take into account realized market conditions through \( f - R^f P \).

### 2.5 Intermediary and Asset Markets Clearing

**Intermediary Market Clearing** Investors can freely allocate capital between the active and passive firms. Since there is a fixed mass of funds of each type available to investors in the two management companies, in equilibrium they must be indifferent in investing their
marginal dollar between these two options. We assume that each fund can only manage $W_0$ in initial capital, and therefore there is decreasing returns to scale at the fund level, as is observed empirically. For the market to then clear, it must be the case that:

$$y_i = \chi.$$  

This free-entry assumption is similar to that in Berk and Green (2004), where the “net fees” of active funds versus passive funds offers similar returns, while “gross of fees” reflects manager skill, and arises because active management skill is in limited supply in our setting.

**Asset Market Clearing** Let $\omega^S_1(i)$ be the portfolio allocation of active fund manager $i \in [0, \chi]$, and similarly with $\omega^D_1$ for passive fund manager $i \in [0, \chi]$. We assume the supply of the asset is given by the vector $\mathbf{x}$ for the $N$ assets. Market-clearing then requires that:

$$\chi \int_0^1 \omega^S_1(i) \, di + (1 - \chi) \omega^D_1 = \mathbf{x}. \quad (4)$$

As is common in the literature, we assume that asset supply $\mathbf{x}$ is noisy to prevent beliefs from being degenerate. We assume that, from the perspective of all agents, $\mathbf{x} \sim \mathcal{N}(\bar{\mathbf{x}}, \tau^{-1}_x \mathbf{I}_N)$ has a multivariate normal distribution. Since all fund managers are atomistic, they take prices as given and each has negligible impact on price formation.

Figure 1 illustrates the time line of the model. We solve for a perfect Bayesian noisy rational expectations equilibrium defined as a list of policy functions $e(C_0^S), \omega^S_1(s_j, \mathbf{P}),$ and $\omega^D_1(\mathbf{P})$, contract $C_0^S$ and prices $\mathbf{P}$ such that: 1) policies solve the respective optimization programs of investors, active fund managers, and active advisory firms; 2) markets clear; 3) agents form expectations according to Bayes’ rule; 4) active manager investment policies are sequential rational. See Appendix A.1 for the detailed definition.
3 The Equilibrium

We search for a symmetric linear equilibrium in which we conjecture that asset prices $P (\Theta, x)$ take the linear form:

$$P (\Theta, x) = \Pi_0 + \Pi_\theta \Theta + \Pi_x x,$$

where $\text{Rank} (\Pi_\theta), \text{Rank} (\Pi_x) = N$. As discussed above, we also focus on linear contracts.

We first derive the conditional beliefs of investors and fund managers. We then derive the optimal investment policy for fund managers, and impose market clearing to solve for equilibrium asset prices. Finally, we solve for the contracts offered by the active advisory firm, and the allocation decision of investors.

3.1 Learning

We begin by deriving the learning process for investors and passive fund managers. Since they have a normal prior, after observing the linear Gaussian signals $P (\Theta)$, they update to a posterior for $\Theta$ that is also Gaussian $\Theta \mid P (\Theta) \sim N \left( \hat{\Theta}, \Omega \right)$ with conditional mean $\hat{\Theta}$ and
conditional variance $\Omega$, given by:

$$
\hat{\Theta} = \Omega \tau_\theta \bar{\Theta} + \Omega \tau_x \Pi'_\theta (\Pi_x \Pi'_x)^{-1} (P - \Pi_0 - \Pi_x \bar{x}),
$$

(6)

$$
\Omega^{-1} = \tau_\theta I_d + \tau_x \Pi'_\theta (\Pi_x \Pi'_x)^{-1} \Pi_\theta.
$$

(7)

Similarly, after observing its vector of private signals, the posterior of active fund manager $j$ is also Gaussian $\Theta | \{P(\Theta), s_j\} \sim \mathcal{N}(\hat{\Theta}(j), \Omega(j))$ with conditional mean $\hat{\Theta}(j)$ and the conditional variance $\Omega(j)$ summarized by the following two expressions:

$$
\hat{\Theta}(j) = \Omega(j) \Omega^{-1} \hat{\Theta} + \Omega(j) \Sigma_j (e_j)^{-1} s_j,
$$

(8)

$$
\Omega(j)^{-1} = \Omega^{-1} + \Sigma_j (e_j)^{-1}.
$$

(9)

In what follows, we denote the conditional risk premium of asset fundamental $\Theta$ as $Z = \hat{\Theta} - R f F^{-1} P$, and $Z \sim \mathcal{N}(\mu, \Omega_Z)$. We define the ex ante mean $\mu$ and variance $\Omega_Z$ in Appendix 2.

### 3.2 Passive Fund Managers

As discussed in the previous section, passive fund managers trade to hold the mean-variance efficient portfolio:

$$
\omega_1^D = \frac{1}{\gamma} (F \Omega F')^{-1} \left( F \hat{\Theta} - R f P \right).
$$

(10)

This portfolio is consistent, for instance, with an optimization over mean-variance or CARA preferences in this normal setting. The superscript $D$ indicates that this is the investment portfolio for passive fund managers given information set $\mathcal{F}_c$.

### 3.3 Active Fund Managers

Since the effort and portfolio choices of active managers are unobservable, they choose incentive compatible portfolios that solve the inner optimization program $\Pi$. Conditional on this
portfolio choice, which has both a mean-variance component and a hedge against the excess payoff portion of their contract, they choose their optimal effort to minimize the conditional variance of their excess payoff. This is summarized in Proposition 1.

**Proposition 1** The optimal portfolio of an active manager $\omega^S$ is given by:

$$
\omega^S (j) = \frac{1}{\gamma_M \rho_S} (F \Omega (j) F')^{-1} (F \hat{\Theta} (j) - R' P) - \frac{1}{\rho_S} \rho_R,
$$

and the optimal level of effort $e$ when the manager’s participation constraint does not bind satisfies:

$$
\text{Diag} \left[ (\Omega^{-1} + \Sigma_j (e_j)^{-1})^{-1} \right] \leq h' (e' 1_{N \times 1}) 1_{N \times 1}, \quad (11)
$$

where $\text{Diag}$ is the diagonal operator. When the participation constraint binds, the optimal effort instead satisfies:

$$
\frac{1}{2} h \left( (e')' 1_{N \times 1} \right) - \frac{1}{2} \log \left| I_{N} + \Sigma_j (e')^{-1} \Omega \right| \\
= \log (\frac{-1}{u_0}) + \gamma_M \rho_0 - \mu' (\Omega + \Omega_Z)^{-1} \mu + \log |\Omega_Z \Omega^{-1} + I_{N}|
$$

From Proposition 1, the linear contract induces the active fund manager to take the optimal mean-variance portfolio given its beliefs, with effective risk aversion $\gamma_M \rho_S$, corrected by a (short) hedging position $-\frac{1}{\rho_S} \rho_R$ that takes into account that the manager is exposed to payoff risks $f - R' P$ independent of the return on the portfolio it manages. This is similar to the findings in the previous literature (i.e., [Brennan (1993)], [Admati and Pfleiderer (1997)], [Buffa et al. (2014)]). The optimal level of effort $e$ sets the marginal benefit of learning, the left-hand side of equation (11), equal to the marginal cost of effort, the right-hand side. Importantly, this marginal benefit is diminishing in how informative are prices, measured by $\Omega^{-1}$, since information in prices acts as a substitute for private information. Since prices are determined by the trading of active managers, the provision of incentives through their
compensation contracts impacts their effort decisions. When instead the participation constraint binds, however, then effort is constrained by the reservation utility $u_0$, and targets the impact of effort on the level of the active manager’s utility rather than its marginal utility. Interestingly, the manager’s fixed fee $\rho_0$ induces more effort in this constrained case.

The correlation structure of asset payoffs $F$ induces substitutability in learning across asset fundamentals $\Theta$ for active fund managers, in addition to the ex post correlation in beliefs captured in $\Omega$. Active fund managers choose their effort recognizing that learning about asset-specific fundamental $\theta_i, i \in \{2, 3, ..., N\}$ also reveals information about the aggregate fundamental $\theta_1$ through prices, which further reveals information about the other asset-specific fundamentals $\theta_j$ for $j \neq i$. In the special case that $F$ is diagonal, the FOC for the optimal effort from Proposition 1 reduces to equation:

$$\frac{1}{\Omega_{ii}^{-1} + M + e_i} \leq h'(e'1_{N \times 1}) \quad \forall \; i \in \{1, ..., N\}.$$ 

The benefit to the active fund manager for increasing effort then becomes separable across assets $\frac{1}{\Omega_{ii}^{-1} + M + e_i}$, and the effort choice for each fundamental is made independent of the others. This substitutability then helps us understand the allocation of efforts across different assets in the performance evaluation.

Having characterized the policies of fund managers, we solve for equilibrium asset prices by imposing market clearing. Appendix A.2 contains the solution of equilibrium asset prices.

### 3.4 Active Fund Contract

We now focus on the contract $C^S_0$ that the active advisory firm offers its fund managers. Proposition 2 provide a characterization of the contract.

**Proposition 2** The contract for an active fund manager is a $N + 2 \times 1$ vector $(\rho_0, \rho_S, \rho_R)$
with the following properties: 1) The optimal choice of \( \rho_R \) is given by:

\[
\rho_R = - \left( \rho_S - \frac{\gamma}{\gamma M} (1 - \rho_S) \right) \omega^0,
\]

where \(- \left( \rho_S - \frac{\gamma}{\gamma M} (1 - \rho_S) \right) \geq 0 \) and:

\[
\omega^0 = \frac{1}{\gamma} F'^{-1} \text{Var} \left( \Theta - R^f F^{-1} P \right)^{-1} E \left[ \Theta - R^f F^{-1} P \right] = \frac{1}{\gamma} F'^{-1} (\Omega_Z + \Omega)^{-1} \mu,
\]

is the ex ante mean-variance efficient portfolio, 2) the optimal sensitivity to the realized excess payoff of the active manager’s fund \( \rho_S \leq \frac{\gamma}{\gamma + \gamma M} \) satisfies the FONC \([A, A]\), with equality when \( y_i = 1 \), and 3) \( \rho_0 \) is given by equation \([A, A]\) and is set such that \( y_i = \chi \), with all equations given in the Appendix.

Proposition 2 reveals that the optimal choice of performance-based incentive \( \rho_S \) is (weakly) below that of perfect risk-sharing \( \frac{\gamma}{\gamma + \gamma M} \). When investors allocate only part of their wealth to active management, \( y_i < 1 \), then the investor and active manager are asymmetrically exposed to the fund-specific risk of noise in the manager’s private information. The manager is exposed to every dollar of the fund’s return, while the investor is exposed to only \( y_i \) of this dollar. It is this departure from perfect risk-sharing \([Admati and Pfeiderer (1997)]\) that introduces a role for benchmarking to help align preferences, since when \( \rho_S = \frac{\gamma}{\gamma + \gamma M} \), then \( \rho_R = 0 \). In our setting, \( \rho_S \) depends on equilibrium prices, and this introduces a contracting externality because contract incentives determine prices.

It is instructive to consider the case, in which we fix \( \rho_S \) at its value under perfect-risk sharing, \( \rho_S = \frac{\gamma}{\gamma + \gamma M} \). Then, \( \rho_R = 0 \), and:

\[
\rho_0 = \frac{(1 - y_i) Tr [A\Sigma_j(e_j)^{-1}]}{\gamma} > 0,
\]

where \( A \) is a matrix given in the Appendix, and \( Tr [A\Sigma_j(e_j)^{-1}] \geq 0 \).

Active managers earn a positive fee to ensure that investors are willing to invest a fraction
\( y_i \) of their wealth in active management. As \( y_i \) approaches 1, investors and active managers become symmetrically exposed to the fund’s return, and the fixed fee vanishes. We are then back in the standard framework in which the affine contract features only performance-based incentives with perfect-risk sharing. Given that the effective risk-aversion of the investor is \( y_i \gamma \), one may instead expect a risk-sharing rule of \( \frac{y_i \gamma}{y_i \gamma + \gamma_M} \), and our equilibrium \( \rho_S \) is indeed closer to this value.

To help further explore the implications of Proposition 2, we rewrite the contract for an active manager as:

\[
C^S_0 = \rho_0 + \rho_S \omega^S_0 (i)' (f - R^f P) - \left( \rho_S - \frac{\gamma}{\gamma_M} (1 - \rho_S) \right) \omega^0_0 (f - R^f P) \\
= \rho_0 + \rho_S (\omega^S_0 (i)' - \omega^0_0) (f - R^f P) + \frac{\gamma}{\gamma_M} (1 - \rho_S) \omega^0_0 (f - R^f P)
\]

The first piece of the contract is a constant fee that ensures that investors are, at the margin, indifferent between investing with active and passive funds. The second piece is the manager’s compensation based on the fund’s performance relative to the ex ante mean-variance portfolio \( \omega^0_0 \). The third adjusts compensation by the performance of an index that tracks this passive portfolio. Consequently, compensation beyond a fixed fee is offered for the value added by the active manager over the investment strategy that investors could achieve through direct investment without acquiring any public or private information.

Benchmarking arises endogenously in the contract offered to active managers. Proposition 2 reveals that the optimal benchmark in our setting is the ex ante mean-variance portfolio formed at date 0, the date that the contract is signed.\(^{16}\) Intuitively, this benchmark represents the outside option to investors at the time of contracting. This benchmark is weighted by a factor \(- \left( \rho_S - \frac{1}{\gamma_M} (1 - \rho_S) \right)\) that aids in risk-sharing by adjusting for the relative risk aversions of the active manager and investor, and is decreasing in \( \rho_S \). This is

\(^{16}\)The ex ante mean-variance portfolio will also be the market portfolio if trading is allowed to occur at \( t = 0 \), since all investors and managers are initially identical. Consequently, one can view the benchmark as the market portfolio in a CAPM world.
similar to the optimal benchmark in van Binsbergen et al. (2008), which features a tilt from the minimum variance portfolio that corrects for differences in risk attitudes between the active fund manager and the delegating CIO. Under certain conditions, this passive portfolio is also featured as an optimal benchmark in Admati and Pfleiderer (1997). Ou-Yang (2003) finds a similar optimal benchmark in the special case when the cost of investing for the manager in their framework is zero, and Dybvig et al. (2010) shows the optimality of the efficient portfolio given the available public information on the market return.

**Comparative Statics** To illustrate how the behavior of active managers and their contract incentives respond to the underlying asset environment, we illustrate our delegated learning mechanism with a two-asset numerical example. We choose as our baseline specification:

\[
F = \begin{bmatrix} 1 & 0 \\ b & \sqrt{1 - b^2} \end{bmatrix}, \quad \bar{\Theta} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \bar{x} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]

In our discussion in this section, we refer to the asset whose payoff depends only on the aggregate fundamental \( \theta_1 \) as Asset 1, and the asset that also has an asset-specific fundamental \( \theta_2 \), with loading \( b \) on \( \theta_1 \), as Asset 2. The \( F \) matrix is set to ensure that the comparative static of \( b \) is implemented keeping the level of uncertainty constant. Finally, we choose the effort function \( h(\cdot) \) to be linear in effort \( \mathbf{e}' \mathbf{1}_{2 \times 1} \), \( h(\mathbf{e}' \mathbf{1}_{2 \times 1}) = \mathbf{e}' \mathbf{1}_{2 \times 1} \), so that the marginal cost of learning is constant. As a result, any substitutability in learning arises from the covariance structure of asset prices.\(^{18}\)

We conduct comparative statics with respect to \( \tau_{\theta} \) and \( b \) to understand how the the behavior of active managers and the two key features of the active manager contract, performance-based incentives \( \rho_s \) and the benchmark portfolio, change with the asset environment. Panels

\(^{17}\)Admati and Pfleiderer (1997) identify the global minimum variance portfolio, tilted by the assets held by investors in separate accounts, as the optimal benchmark in a partial equilibrium setting. Since prices are endogenous, our benchmark portfolio is, itself, an equilibrium object that depends on the contract.

\(^{18}\)Although we consider a two-asset example for ease of exposition, we find that our results hold more generically.
(a) and (c) in Figure 2 show that as the ex ante uncertainty of asset payoffs $\tau_\theta^{-1}$ declines, there is less benefit for active managers in acquiring information. As a result, their contracts put more weight on their performance-based incentives by increasing $\rho_s$, which attenuates the aggressiveness with which they trade by making them effectively more risk-averse. This, in equilibrium, reduces the information content in prices and, since private information is a substitute for public information, incentivizes active managers to exert more effort.

Panels (a) and (c) of Figure 3 reveal that the holdings of active managers and their benchmark portfolio respond in opposite directions to an increase in $\tau_\theta$. As $\tau_\theta$ increases, the payoffs of risky assets are less uncertain, and the representation of these assets in the benchmark portfolio increases. This occurs because the benchmark is determined based on the assets’ ex ante risk-return tradeoff, and this risk is decreasing. Since active managers’ compensation is tied to the benchmark portfolio, they hedge this risk by taking a long position in it. This hedging demand, which is also present in, for instance, Cuoco and Kaniel (2011), Leippold and Rohner (2012), and Buffa et al. (2014), drives up the equilibrium prices of assets in the benchmark, which benefits active managers by lowering the benchmark portfolio’s expected payoff. As their effort and the value of private information falls, active managers trade less on their private information, reducing their active positions as their holdings are driven more by their hedging demand. As uncertainty declines, consequently, the holdings of active managers approach that of the benchmark.

Varying the correlation structure of asset payoffs also impacts the behavior of active managers and the incentives they face. As shown in Panels (b) and (d) of Figure 2, as $b$ increases, asset payoffs become more correlated with the aggregate component. As a result, the marginal benefit of learning about asset-specific information decreases, while learning about the aggregate component has a higher, though not off-setting, marginal benefit, leading to an overall decline in manager effort. Effort to learn about the aggregate component increases less than the effort to learn about the asset-specific component decreases because prices are also more revealing about this aggregate component, as the payoffs of both assets
become driven more by it. Similar to an increase in $\tau_\theta$, the active advisory firm responds to the increase in correlation, which causes active manager effort to fall and prices to become more informative, by increasing $\rho_S$.

**Figure 2: Performance-based Compensation and Optimal Effort**

Parameters: $\tau_\theta = 0.5$, $\tau_x = 1$, $b = 0.5$, $\gamma_M = 2$, $M = 0$, $\chi = 0.3$, $\gamma = 1$, $W_0 = 1$, $R_f = 1.02$

Panels (b) and (d) of Figure 3 change of correlation structure $b$ impacts the holdings of active managers and their benchmark portfolio as well. As $b$ increases, and the two assets become more correlated, there is less diversification benefit in holding a portfolio of the two assets. As a result, the holdings of both assets decline in the benchmark. The response of the holdings of active managers to the increase in correlation again reflects both their change in effort and the change in the composition of the benchmark. As they shift their effort toward learning more about the aggregate component of asset payoffs, their holdings become more tilted toward asset 1, which is driven more by the aggregate component than asset 2. In
addition, as the benchmark portfolio reduces its holdings, the active managers take longer position to reduce their hedging demand.

The tension between the speculative position taken by active managers based on their private information and their hedging demand for the benchmark portfolio is one of the central focuses of our analysis. While the former represents active management skill in our framework, as it reflects the value that active managers add in portfolio selection, the latter reflects a response to risk-sharing incentives in their compensation contracts. Both are sources of variation in the holdings of active managers, yet the two do not always move together with changes in the underlying asset environment. A similar tension arises between the speculative position of active managers and the asset risk-return tradeoff, as effort and risk premia can move in opposite directions with changes in the underlying asset environment. Such tensions will be important in our discussion of the consistency of performance evaluation metrics.

### 3.5 Contract Externality

In this subsection, we highlight a contract externality that arises because prices are determined in equilibrium by active managers whose trading is governed by their contract incentives. Since the weight the active fund contract places on the fund’s performance, $\rho_S$, determines each active manager’s effective risk aversion, it impacts how aggressively they trade on their private information and, consequently, the information content of prices and the risk-return tradeoff. Through this channel, contract incentives at an industry level impact not only the effort that each active manager exerts, but also the choice of performance-based and benchmarking incentives the active advisory firm offers each manager. This latter effect is absent when there is perfect risk-sharing between investors and active managers, as $\rho_S$ is fixed only by their risk aversions. In what follows, we refer to the $\rho_S$ that is reflected in prices as the industry incentives, $\hat{\rho}_S$, and the $\rho_S$ of each manager as that manager’s $\rho_S$.

The first aspect of this externality, the impact of industry incentives on effort, is sum-
Figure 3: Active Holdings and Benchmark Portfolio

Parameters: $\tau_0 = 0.5$, $\tau_x = 1$, $b = 0.5$, $\gamma_M = 2$, $M = 0$, $\chi = 0.3$, $\gamma = 1$, $W_0 = 1$, $R^f = 1.02$

Proposition 3
The optimal choice of active fund manager effort $e_k$, is increasing (element-by-element) in the industry sensitivity of active manager compensation to the realized fund’s performance, $\hat\rho_S$. It is decreasing in the fraction of active fund managers, $\chi$.

From Proposition 3, the industry sensitivity of active managers’ compensation to their fund’s performance increases the effort that each active fund manager exerts to learn about asset payoffs. While the sensitivity of active managers’ compensation to their fund’s performance typically aids in preference alignment in standard principal-agent problems, it serves a dual role in our setting in incentivizing effort through its impact on prices. Similarly, the size of the active management industry, which determines the extent to which $\hat\rho_S$ is reflected in prices, also impacts active manager effort. The smaller the size of the active management
industry, the less information is incorporated into prices. Since active managers substitute between public and private information, the less informative are prices, the more effort that each exerts.

In addition to impacting effort, the industry sensitivity of active managers’ compensation to their fund’s performance $\hat{\rho}_S$ also feeds back into the optimal choice of $\rho_S$ for each manager, similar to García and Vanden (2009) in the case of manager fees. Our analysis highlights that such a price setting externality also arises in a principal-agent setting when investors invest in both active and passive vehicles, and influences performance-based incentives in compensation contracts. Figure 4 illustrates that this externality is negative in our numerical example, across both $\tau_\theta$ and $b$, and highlights that there is substitutability in contract incentives. The stronger the industry’s incentives for active management, $\hat{\rho}_S$, the weaker the incentives for each individual active manager, $\rho_S$, and this effect transmits through prices. By reducing how informative are prices, a higher $\hat{\rho}_S$ increases the benefit of trading on private information, and this is reflected in a smaller $\rho_S$ that induces active manager’s to trade more aggressively. Interestingly, this substitutability becomes stronger as overall uncertainty ($\tau_\theta$) increases, while it becomes weaker as payoff correlation ($b$) increases.

While benchmarking in active management entails a price externality, since the aggregate hedging demand of active managers lowers the expected return of the benchmark’s
constituents, our setting highlights that there is also a contracting externality that feeds back into the benchmark component of active manager incentives. Since the industry sensitivity to performance \( \hat{\rho}_S \) impacts the sensitivity \( \rho_S \) that each manager faces, it follows that \( \hat{\rho}_S \) also affects the weight that the compensation of active managers places on the benchmark portfolio, \(- \left( \rho_S - \frac{\gamma}{\gamma_M} (1 - \rho_S) \right) \geq 0 \). Since each manager’s \( \rho_S \) is decreasing in \( \hat{\rho}_S \), it follows that this weight becomes larger as \( \hat{\rho}_S \) increases, and consequently the active contract puts more weight on the realized payoff of the benchmark portfolio.

4 Performance Evaluation

In this section, we explore our model’s implications for the performance evaluation of active managers. We first construct theoretical analogues of three commonly employed empirical measures of manager skill, active share, return gap, and expected excess returns, and investigate to what extent they consistently reflect the skill of active managers in our model. An advantage of our equilibrium approach is that we can relate characteristics of the assets in which active funds invest to potentially observable outcomes, such as their holdings and performance, without needing to observe their compensation contracts or proxies for their (unobservable) effort to acquire private information. We then provide an explanation for the recent decline in fund flows to the active management industry in the context of our delegated learning framework.

4.1 Performance Evaluation

In what follows, we consider the value that active managers add to portfolio selection to be the private information that they acquire. We define our model-implied measure of skill as the reduction in that manager’s uncertainty about asset payoffs, \( |\Omega| - |\Omega(j)| \). We next construct theoretical analogues of three commonly employed empirical measures of skill, \(^{19}\) Our results are qualitatively similar if we use other model-implied measures of skill, such as \( e_j \) or \( |\Omega - \Omega(j)| \).

\(^{19}\)Our results are qualitatively similar if we use other model-implied measures of skill, such as \( e_j \) or \( |\Omega - \Omega(j)| \).
the active share introduced by Cremers and Petajisto (2009), the return gap of Kacperczyk et al. (2008), and expected excess returns, and examine how well they correlate with our model-implied measure as we vary the asset environment.

To relate the effort exerted by active managers to the active share introduced by Cremers and Petajisto (2009), we define our active share as the deviation of an active fund manager’s portfolio holdings from the benchmark portfolio.

We derive an analogous expression for the average active share of an active fund manager in our economy \( AS \):

\[
AS = \frac{1}{2} E \left[ 1' \left| \omega_1^S (j) - \omega^0 \right| \right],
\]

where \( \omega^0 \) is the benchmark for the active manager.\(^{20}\) Substituting for \( \omega_1^S (j) \) with equation (1), \( \omega^0 \) with Proposition 2, \( \hat{\Theta} (j) \) with equation (8), and \( \Pi_\theta \) and \( \Pi_x \) with equations (A2) and (A3), respectively, we can employ results for the expectation of a folded normal distribution to arrive at:

\[
AS = \frac{1}{2\gamma M\rho S} \sum_{i=1}^{N} \left( \sqrt{\frac{2}{\pi}} \sigma_i \varepsilon^{\mu_i^2 / 2\sigma_i^2} + \mu_i \left( 1 - 2\Phi \left( -\frac{\mu_i}{\sigma_i} \right) \right) \right),
\]

where:

\[
\mu_i = f'_i \left( \Omega^{-1} + \Sigma_j (e_j)^{-1} - (1 - \rho_S) (\Omega_Z + \Omega)^{-1} \right) \mu,
\]

\[
\sigma_i^2 = f'_i \left( \Gamma_\theta \tau_\theta^{-1} \Gamma'_\theta + \Gamma_x \tau_x^{-1} \Gamma_x + \Sigma_j (e_j)^{-1} \right) f_i,
\]

where \( f_i \) is the \( i^{th} \) column of \( F^{-1} \), and

\[
\Gamma_\theta = \tau_x \left( \frac{\chi}{\gamma M\rho S} \right)^2 \Sigma_j (e_j)^{-1} (F'F)^{-1} \Sigma_j (e_j)^{-1} - R_f^f \left( \Omega^{-1} + \Sigma_j (e_j)^{-1} \right) F^{-1} \Pi_\theta + \Sigma_j (e_j)^{-1},
\]

\[
\Gamma_x = \left( \tau_x \Sigma_j (e_j)^{-1} (F'F)^{-1} \Sigma_j (e_j)^{-1} - R_f^f \left( \frac{\gamma M\rho S}{\chi} \right)^2 \left( \Omega^{-1} + \Sigma_j (e_j)^{-1} \right) F^{-1} \Pi_\theta \right) \Sigma_j (e_j) F'.
\]

\(^{20}\)One may notice that the definition of active share includes fund leverage. Since the benchmark can also take leveraged positions, this ensures an equitable comparison of portfolios when measuring manager activeness.
Figure 5 shows the comparative statics of our model-implied measure of skill and our theoretical analogue of active share with respect to the overall uncertainty and the correlation of asset payoffs, $\tau_\theta$ and $b$, respectively. As shown in panels (b) and (d) of Figure 2, active managers exert less effort to acquire private information as the uncertainty of asset payoffs decreases ($\tau_\theta$ increases) or as the correlation of asset payoffs $b$ increases. This is reflected in our model-implied measure of skill, which falls with both $\tau_\theta$ and $b$, as shown in panels (b) and (d) of Figure 5.

Figure 5 also illustrates how our analogue of active share varies with the asset environment. As shown in panels (a) and (c) of Figure 3, as $\tau_\theta$ and $b$ increase, the holdings of active funds reflect that active managers are acquiring less private information. As overall uncertainty $\tau_\theta^{-1}$ falls, active managers tilt their portfolio more toward the benchmark portfolio, reflecting the decline in their informational advantage. As payoff correlation $b$ increases, however, they instead tilt their portfolio away from the benchmark, reflecting both a smaller (short) hedging position and a more active tilt towards asset 1 from acquiring marginally more private information. As a result, while our theoretical analogue of active share falls with respect to $\tau_\theta$ in panel (a) of Figure 5, it instead increases in panel (d) with respect to $b$. Consequently, while our analogue of active share correctly captures the decline in skill with respect to overall uncertainty, it gives an inconsistent prediction with respect to correlation. This inconsistency reflects the tension discussed in Section 3 between the speculative position taken by active managers based on their private information and their hedging demand for the benchmark portfolio.

Our prediction on the relation between a fund’s active share and its benchmark also echoes the critique of Frazzini et al. (2016). Active share can deviate from the underlying level of active manager skill, since the incentives for the active manager to acquire private information are shaped by the asset environment, which is reflected in its benchmark. While funds that invest in more volatile stocks are and appear more active, funds that invest in more correlated stocks may only appear more active because active share captures the decline in
Figure 5: Active Share and Active Fund Manager Skill: $\tau_\theta$ & $b$

Parameters: $\tau_\theta = 0.5$, $\tau_x = 1$, $b = 0.5$, $\gamma_M = 2$, $M = 0$, $\chi = 0.3$, $\gamma = 1$, $W_0 = 1$, $R_f = 1.02$
the hedging demand for the benchmark portfolio rather than an increase in skill. That active managers are more active from learning when payoffs are more uncertain is consistent with Jiang and Sun (2014), who finds that the dispersion in fund managers’ beliefs, based on the dispersion in their active holdings, is positively correlated with the idiosyncratic volatility of the stocks in which they invest.

Our setting also allows us to explore another empirical measure of fund manager skill, the return gap ($RG$) employed in Kacperczyk et al. (2008). First, we can rewrite the portfolio of each active manager as:

$$\omega^S_1(j) = \frac{\gamma}{\gamma_M \rho_S} \omega_1^D + \left(1 - \frac{\gamma}{\gamma_M} \left(\frac{1}{\rho_S} - 1\right)\right) \omega_0 + \frac{1}{\gamma_M \rho_S} F' \Sigma_j (e_j)^{-1} \left(s_j - R^F F^{-1} \mathbf{P}\right).$$

The first two elements reflect the position an active manager without private information would take based on public information and the benchmark portfolio, while the last element captures the speculative bet it makes based on its informational advantage after observing its private signals. Consequently, we can view the first two elements as the “holdings” portfolio that is publicly observable to investors, and measure the expected return gap between the gross return an active manager garners and that of this “holdings” portfolio, $RG$. We can then construct the expected return gap as:

$$E[RG] = \frac{1}{\gamma_M \rho_S} \mu' \Sigma_j (e_j)^{-1} \mu + \frac{1}{\gamma_M \rho_S} Tr \left[\Sigma_j (e_j)^{-1} (\Omega_Z + \Omega)\right].$$

The expected return gap is driven by 1) active managers trading more aggressively to collect the risk premia on assets since they face less risk because of their private information, the first term, and 2) the reduction in overall uncertainty they have when speculating, the second term.

Figure 6 shows the comparative statics of the expected return gap and the model-implied measure of skill. The return gap is moving in the same direction as the model-implied measure of skill when the overall uncertainty of payoffs $\tau^{-1}_0$ declines. This reflects that, as
uncertainty falls, both the informational advantage of active managers and the excess return to their risk-taking deteriorate. The return gap, however, is hump-shaped in the correlation of asset payoffs, which reflects two competing forces. On the one hand, there is increased risk in asset markets because a higher correlation among asset returns reduces the diversification benefit to holding both assets. Since they have access to private information, active fund managers take larger exposures to the risky assets than passive managers, and earn higher returns from the higher risk premia. On the other hand, the increased correlation also reveals more information to the passive managers about the aggregate asset fundamental, reducing the information asymmetry between active and passive managers. These two forces contribute to the humped-shaped return gap in Panel (c) in Figure 6. This inconsistency reflects the tension discussed in Section 3 between the speculative position taken by active managers based on their private information and the risk-return tradeoff of the asset environment.

Finally, we can evaluate fund manager performance by computing the expected excess returns for the benchmark portfolio, and for the holdings of both active and passive managers. Making use of properties of chi-squared random variables and the trace operator, we arrive at the expected excess return of active managers, net of fees:

\[
E \left[ W^S_2 - C^S_0 - R^f W_0 \right] = \left( \rho_S - \frac{\gamma}{\gamma^M} (1 - \rho_S) \right) E \left[ W^0_2 - R^f W_0 \right] - \rho_0 \\
+ \frac{\gamma}{\gamma^M} \left( \frac{1}{\rho_S} - 1 \right) E \left[ W^D_2 - R^f W_0 \right] + (1 - \rho_S) E \left[ R G \right],
\]

as well as for passive managers:

\[
E \left[ W^D_2 - R^f W_0 \right] = \frac{1}{\gamma^M} \mu^T \Omega^{-1} \mu + \frac{1}{\gamma} Tr \left[ \Omega^{-1} \Omega_Z \right],
\]
Figure 6: **Return Gap and Fund Manager Skill: $\tau_{\theta}$ & $b$**

Parameters: $\tau_{\theta} = 0.5, \tau_{x} = 1, b = 0.5, \gamma_{M} = 2, M = 0, \chi = 0.3, \gamma = 1, W_0 = 1, R^f = 1.02$

and the benchmark portfolio:

$$
E \left[ W_2^0 - R^f W_0 \right] = \frac{1}{\gamma} \mu' (\Omega_Z + \Omega)^{-1} \mu.
$$

The expected excess return of active managers, net of the fixed fee $\rho_0$, can be decomposed into three parts: 1) a return proportional to the benchmark portfolio, 2) a return proportional to the return of passive managers, and 3) the expected return gap.

As asset payoff uncertainty falls, the expected excess return of both passive managers and the benchmark portfolio falls, reflecting the decline in risk in the market. In addition, the expected return gap also falls, from panel (a) in Figure 6 and consequently the expected excess return of active managers falls. While the expected excess returns of passive funds and the benchmark portfolio increase as correlation increases, reflecting greater risk, expected
return gap is hump-shaped from panel (c) in Figure 6. This suggests, similar to active share, that the expected excess return of active managers can also be an inconsistent measure of active manager skill.

4.2 Decline in Fund Flow and Performance in Active Management

Since 2006, over 90% of U.S. actively managed equity funds failed to beat their benchmark net of fees. During this time, they have also lost fund flows to passive strategies, both domestically and globally. A potential explanation for these phenomena is that there has been a downward trend in the level of skill among active asset managers. As can be seen in (Figure 5 (d)), our model predicts that a higher (pairwise) correlation between assets reduces the effort that active managers exert to acquire private information. Cotter et al. (2016), among others, document a pronounced increase in the level of integration within and among assets classes, and across countries, since the 2008 financial crisis. Such an increase in asset correlations can, consequently, explain why active managers have under-performed in recent years, if their compensation contracts did not adjust to the new asset environment. Consistent with this view, much of the recent debate has been about reforming the incentive structure for the asset management industry.

5 Measuring Skill: The Dynamic Extension

In this section, we discuss a dynamic extension of our model in which active managers trade over multiple periods. Further details of the model, its derivation, and a more thorough discussion of the results are in the Internet Appendix. In what follows, we describe the salient features of the analysis.

We use the multi-period model to generalize several of our insights to a dynamic setting. As is endemic to dynamic portfolio choice problems, the portfolio allocation decisions of active and passive managers now also include an intertemporal hedging motive to insure
against future fluctuations in the payoff environment. Novel to our setting is that the effort choice of active managers is now also forward-looking. Whereas in the static setting, active managers seek to minimize the conditional variance of their portfolio excess payoff, in the dynamic setting managers also take into account the benefits of learning in early versus later periods. The active advisory firm internalizes these intertemporal incentives when choosing the optimal affine contract to offer its managers, and the optimal benchmark portfolio generalizes the notion of the ex ante mean-variance portfolio to take into account the average expected changes in the means and variances of the investment opportunity set.

A central point of departure in the multi-period setting is the possibility that investors can observe a time series of active manager performance during the intermediate trading periods. In practice, the SEC N-Q and N-CSR filings, which are publicly available, require large mutual funds to report all long positions held at the end of a quarter. This potentially enables investors to improve their monitoring of fund managers’ behavior. Based on the availability of the SEC N-Q and N-CSR filings for mutual funds, we allow investors to observe an unbiased but noisy measure of their active fund’s return gap at each date $t$. This noise, which we assume is i.i.d. across assets and dates, can be thought of as portfolio rebalancing driven by non-fundamental, non-informational reasons or measurement error. Let $R_t^i$ be this noisy observation. Given their observations of past asset and fund-specific returns, investors can form their posterior beliefs about their fund manager’s private information conditional on a given path of effort $\{e_t(i)\}_{s=1}^T$. They can then derive a log-likelihood ratio under the null hypothesis that their active manager exerts no effort $H_0 : \{e_t(i)\}_{s=1}^T = \{0_{N \times 1}\}_{s=1}^T$, which corresponds to no exhibition of ability. We show that the weighted historical variance of this return gap $S_T$ is related to the log-likelihood ratio of the null hypothesis of no effort to an alternative hypothesis of some schedule of effort:

$$S_T = \frac{1}{T} \sum_{t=1}^T \left( \frac{(R_t^i)^2}{Var_U[R_t^i | \mathcal{F}_t]} - \frac{(R_t^i - E[R_t^i | \mathcal{F}_t^I(i)])^2}{Var[R_t^i | \mathcal{F}_t^I(i)]} \right).$$
and that it is a consistent estimator of whether the active manager exhibits skill. In the above statistic, $\text{Var}_{U} [R^I_t \mid \mathcal{F}^I_t]$ is the conditional variance of the return gap if the manager exerts no effort to learn or has no ability, $E [R^I_t \mid \mathcal{F}^I_{t-1} (i)]$ is the best predictor of the return gap given all public information available to investors, including current realized asset returns, and past fund returns, and $\text{Var} [R^I_t \mid \mathcal{F}^I_t (i)]$ is the corresponding conditional variance.

While Admati and Pfleiderer (1997) demonstrate that the first moment of benchmark-adjusted returns is not sufficient to identify skill, our analysis suggests that investors should focus on second moments. Intuitively, in any given trading period, an active manager may appear more or less active because of noise in their information or time variation in expected returns: however, systematically their portfolios should deviate from their passive counterparts. This motivates examining dynamic measures of skill, such as the empirical analogue of our $S_T$ statistic, to measure skill in an active management.

An important feature of the mutual fund industry is the sensitivity of fund flows to past returns. Such flow-performance sensitivity provides fund managers with an indirect incentive to grow their fund’s AUM through career concerns. Our dynamic model provides a microfoundation for this sensitivity. If investors allocate capital to funds based on the historical $S_T$ statistic, then managers face forward-looking incentives for effort to achieve a higher $S_T$ statistic. To see this, suppose that a new generation of investors at date $T$ cannot distinguish between active and passive managers, and allocates their capital to funds in proportion to the signal that the fund manager has skill based on the fund’s returns, according to the rule $w (S_T)$:

$$w (S_T) = (S_T - S_{crit, \alpha}) W_0,$$

where $\frac{S_{crit, \alpha}}{\sigma_0} = \Phi^{-1} (1 - \alpha)$ is the $\alpha$–level of confidence under the null hypothesis, and $\Phi (\cdot)$ is the CDF of the normal distribution. Interestingly, $w (S_T)$ is increasing and convex in the

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21 We assume implicitly that $T$ is large enough that the asymptotic distribution is a reasonable approximation. Under the null hypothesis of no skill, or $R^I_t \sim iid (0, \text{Var}_{U} [R^I_t \mid \mathcal{F}^I_t])$, $S_T$ has a normal asymptotic
most recent fund return gap $R_T^i$ through $S_T$, leading to a short-term convex flow-performance relation. If the compensation of fund managers is based on their final AUM at date $T$, cum of new fund inflows, then the future flow-performance sensitivity can incentivize forward-looking active managers to exert effort before date $t$ to raise their fund flows at date $T$.

Such a mechanism suggests that convex flow-performance sensitivity may be a reaction to learning about a manager’s skill, and nonlinear flow-performance sensitivity could be a tool for completing the contracting space between investors and funds, which is restricted to be linear by the SEC. Since such implicit compensation schemes can incentivize managers to exert effort by relaxing their IC constraints, investors may be willing to provide such rewards even though there is no direct benefit to them in attracting future investors.

### 6 Conclusion

We study an economy in which investors delegate their capital in financial markets to fund managers, and must incentivize active fund managers to exert costly effort to acquire private information about asset payoffs. Our equilibrium analysis features a novel channel by which performance-based incentives feed into the information acquisition decisions of active managers by impacting the informativeness of asset prices. This allows us to study the rich interaction between compensation incentives in the active management industry, in which benchmarking arises endogenously in our framework, and the learning and trading decisions of active managers. Our model cautions the use of existing empirical measures of skill employed in the literature, and offers a new measure that is motivated by a dynamic extension of our model. Our analysis highlights that the skill of active managers is endogenous to both the asset environment and the compensation incentives they face, and offers a potential explanation for the recent shift in fund flows to passive strategies.

distribution $\mathcal{N}(0, \sigma_0^2)$, where $\sigma_0^2$ is the variance of $S_T$ when $R_T^i$ is i.i.d.
References


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A Appendix

A.1 Equilibrium

A perfect Bayesian noisy rational expectations equilibrium in this economy is a list of policy functions \( y_i, e(C^S_0), \omega^S_{s_j, P} \), and \( \omega^D_{P} \), compensation contract \( C^S_0 \) for fund managers, and prices \( P \) such that:

- **Active Fund Manager Optimization:** Given contract \( C^S_0 \), prices \( P \), and information set \( F_j \), \( e(C^S_0) \), and \( \omega^S_{s_j, P} \) solve each fund manager’s IC constraint.

- **Active Advisory Firm Optimization:** Given \( y_i \), contract \( C^S_0 \) solves the investor’s optimization problem (2) and delivers expected utility \( V_0 \).

- **Investor Optimization:** Given contract \( C^S_0 \), allocation \( y_i \) solves the investor’s optimization problem (2) and delivers expected utility \( V_0 \).

- **Market Clearing:** The intermediary market clears through \( y_i = \chi \) and the asset markets clear through equation (4).

- **Consistency:** Investors and passive fund managers form their expectations about \( \Theta \) based on their information set \( F^c \), while active fund managers form their expectations based on their information set \( F_j \), according to Bayes’ rule.

- **Sequential Rationality:** For each realization of prices \( P \) and private signals \( s_j \), passive and active fund managers find it optimal at date 1 to follow investment policy \( \omega^S_{s_j, P} \) and \( \omega^D_{P} \), respectively.

A.2 Equilibrium Asset Prices

Given the asset demand of direct investors and fund managers from equation (10) and Proposition[4], respectively, we are now in a position to derive equilibrium prices. Aggregating the demand of fund managers and direct investors, \( \omega^S_{1} \) and \( \omega^D_{1} \), respectively, the market-clearing condition reveals that:

\[
\chi \frac{1}{\gamma_{MPS}} (F\Omega(j)F')^{-1} \left( F \int_0^1 \hat{\Theta}(j) \, di - R'P \right) - \chi \frac{1}{\rho_R}(1 - \chi) \frac{1}{\gamma} (F\Omega F')^{-1} (F\hat{\Theta} - R'P) = x.
\]

Substituting for \( \hat{\Theta} \) and \( \hat{\Theta}(j) \) with equations (6) and (8), respectively, and imposing the Strong LLN, we find that:

42
$$\mathbf{P} = \left( \frac{\chi}{\gamma_{M\rho S}} \Omega (j)^{-1} + \frac{1 - \chi}{\gamma} \Omega^{-1} \right) R' F^{-1} - \left( \frac{1 - \chi}{\gamma} + \frac{\chi}{\gamma_{M\rho S}} \right) \tau_x \Pi_0 \left( \Pi_x \Pi'_x \right)^{-1} \right)^{-1} \times \left( \frac{1 - \chi}{\gamma} + \frac{\chi}{\gamma_{M\rho S}} \right) \left( \tau_0 \bar{\Theta} - \tau_x \Pi_0' \left( \Pi_x + \Pi_x \mathbf{x} \right) \right) + \frac{\chi}{\gamma_{M\rho S}} \Sigma_j (e_j)^{-1} \left( 1 + \frac{1 - \chi}{\gamma} \right) \Omega^{-1} + \tau_x \Pi_0 \left( \Pi_x \Pi'_x \right)^{-1} \mathbf{F} - \frac{\chi}{\rho S} \mathbf{R} \right).$$

Matching coefficients with the conjectured form of prices \[\text{(3)}\], and the imposing equation \[\text{(7)}\], we find that:

$$\Omega^{-1} = \tau_0 I d_N + \tau_x \left( \frac{\chi}{\gamma_{M\rho S}} \right)^2 \Sigma_j (e_j)^{-1} (F' F)^{-1} \Sigma_j (e_j)^{-1}, \quad (A1)$$

and that \( \Pi_\theta, \Pi_x, \Pi_0 \) are given by:

\[
\begin{align*}
\Pi_\theta &= \frac{1}{R'} F \left( \tau_0 \left( \frac{\chi}{\gamma_{M\rho S}} \right)^2 \Sigma_j (e_j)^{-1} (F' F)^{-1} \Sigma_j (e_j)^{-1} + 1 + \frac{1 - \chi}{\gamma} \right)^{-1} - \frac{\gamma_{M\rho S}}{\chi} \Pi_0 \Sigma_j (e_j)^{-1} \mathbf{F}' \\
\Pi_x &= -\frac{\gamma_{M\rho S}}{\chi} \Pi_0 \Sigma_j (e_j)^{-1} \mathbf{F}' \\
\Pi_0 &= \frac{1}{R'} F \left( \frac{\chi}{\gamma_{M\rho S}} \Omega (j)^{-1} + \frac{1 - \chi}{\gamma} \Omega^{-1} \right)^{-1} \left( \frac{1 - \chi}{\gamma} + \frac{\chi}{\gamma_{M\rho S}} \right) \left( \tau_0 \bar{\Theta} - \tau_x \Pi_0' \Pi_x' \mathbf{x} \right) - \frac{\chi}{\rho S} \mathbf{F}' \mathbf{R} 
\end{align*}
\]

which confirms the conjectured linear equilibrium.

Several features of the equilibrium are immediately apparent from the price coefficients. We see, for instance, that if \( \Sigma_j (e_j)^{-1} \) is zero, so that fund managers have no private information, then \( \Pi_\theta, \Pi_x \to 0_{N \times N} \), and prices reflect only prior information about the risky asset payoffs. In addition, the signal-to-noise ratio of prices as signals about the risky asset payoffs, \( \Pi_x^{-1} \Pi_\theta = -\frac{\chi}{\gamma_{M\rho S}} F^{-1} \Sigma_j (e_j)^{-1} \), depends not only on the correlation structure of asset payoffs and the effort exerted by fund managers, but also negatively on their risk aversion \( \gamma_M \) and the sensitivity of their compensation to the realized return of their fund, \( \rho_S \). That these latter two features enter as \( \gamma_{M\rho S} \) highlights that \( \rho_S \) makes the fund manager effectively more risk-averse over his fund’s performance, and, as a result, more conservative in his investment policies.
A.3 Proof of Proposition 1

Assuming the linear contract, the IC constraint of the fund manager, conditional on an effort choice \( e \), reduces to the mean-variance optimization problem:

\[
\sup_{\omega_S^i (j)} \left\{ \rho_0 + \rho_s \omega_S^i (j)' \left( F \hat{\Theta} (j) - R' P \right) + \rho_R' \left( F \hat{\Theta} (j) - R' P \right) \right\},
\]

given its CARA-normal structure. It then follows from the FOC for \( \omega_S^i \) at interior solution that:

\[
\omega_S^i (j) = \frac{1}{\gamma M \rho S} (F \Omega (j) F')^{-1} \left( F \hat{\Theta} (j) - R' P \right) - \frac{1}{\rho S} \rho R.
\]

Substituting this optimal portfolio choice into the manager’s utility, the IC constraint when choosing effort level \( e \) becomes:

\[
e \in \arg \sup_{e \in \mathbb{R}^N} \left\{ E \left[ \exp \left( -\frac{1}{2} \left( \hat{\Theta} (j) - R' F^{-1} P \right)' \Omega (j)^{-1} \left( \hat{\Theta} (j) - R' F^{-1} P \right) \right) \right] \right\}.
\]

To solve for the optimal level of effort for fund managers, we invoke the law of iterated expectations and first find the expected utility of a fund manager conditional on having observed market prices. The optimal choice of effort conditional on having observed market prices is independent of the specific realization of prices. As a result, the optimal effort of fund managers conditional on observing prices is also a measurable strategy for fund managers before observing prices. Since unconditional strategies cannot improve on strategies that condition on more information, this optimal effort ex-post must also be optimal ex-ante.

Recognizing that \( s (j) \mid \mathcal{F}_0^c \sim \mathcal{N} \left( \hat{\Theta}, \Omega + \Sigma_j (e_j) \right) \), and that

\[
\hat{\Theta} (j) - R' F^{-1} P = \hat{\Theta} - R' F^{-1} P + \Omega (j) \Sigma_j (e_j)^{-1} \left( s (j) - \hat{\Theta} \right),
\]

where

\[
\Omega (j)^{-1} = \Omega^{-1} + \Sigma_j (e_j)^{-1},
\]

by completing the square for normal random variables, the expected utility of fund manager \( i \) given only the market beliefs and effort \( e' E \left[ \sup_{\omega \in \mathbb{R}^N} E \left[ u \left( C^S_0 ; \omega', e' \right) \mid \mathcal{F}_j \right] \mid \mathcal{F}^c \right] \) is
A minimization program for the effort of fund manager is then equivalent to:

\[
E \left[ \sup_{\omega \in \mathbb{R}^N} E \left[ u \left( C_0^S; \omega', \mathbf{e}' \right) \mid \mathcal{F}_j \right] \mid \mathcal{F}_0^c \right]
\]

\[
= -E \left[ \exp \left( -\frac{1}{2} \left( \hat{\Theta} (j) - R^j F^{-1} \mathbf{P} \right)' \Omega (j)^{-1} \left( \hat{\Theta} (j) - R^j F^{-1} \mathbf{P} \right) \right) \mid \mathcal{F}_0^c \right]
\]

\[
= -\frac{(2\pi)^{\frac{-N}{2}}}{|\Omega + \Sigma_j (\mathbf{e}_j)|^{1/2}} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \left( \hat{\Theta} (j) - R^j F^{-1} \mathbf{P} \right)' \Omega (j)^{-1} \left( \hat{\Theta} (j) - R^j F^{-1} \mathbf{P} \right) \right) ds (j)
\]

\[
= -\exp \left( -\frac{1}{2} \left( \hat{\Theta} (j) - R^j F^{-1} \mathbf{P} \right)' \Omega (j)^{-1} \left( \hat{\Theta} (j) - R^j F^{-1} \mathbf{P} \right) \right)
\]

A similar result can be found by applying results for the moment-generating function of the non-central chi-square random variables. As one can see, the optimal choice of effort enters the conditional expected utility only through the \(-\frac{1}{2} \log |Id_N + \Sigma_j (\mathbf{e}_j)^{-1} \Omega | - \gamma_{MP0} \) term. Since fund managers are price-takers and the conditional variance of market beliefs \( \Omega \) is known ex-ante, we find that:

\[
E \left[ \sup_{\omega \in \mathbb{R}^N} E \left[ u \left( C_0^S; \omega', \mathbf{e}' \right) \right] \right] = -\exp \left( -\frac{1}{2} \left( \hat{\Theta} (j) - R^j F^{-1} \mathbf{P} \right)' \Omega (j)^{-1} \left( \hat{\Theta} (j) - R^j F^{-1} \mathbf{P} \right) \right) 
\]

\[
\times E \left[ \exp \left( -\frac{1}{2} \left( \hat{\Theta} (j) - R^j F^{-1} \mathbf{P} \right)' \Omega (j)^{-1} \left( \hat{\Theta} (j) - R^j F^{-1} \mathbf{P} \right) \right) \right].
\]

We first consider the case when the participation constraint does not bind. The optimization program for the effort of fund manager is then equivalent to:

\[
\mathbf{e} \in \arg\sup_{\mathbf{e}' \in \mathbb{R}_N^+} \left\{ \log |\Omega^{-1} + \Sigma_j (\mathbf{e}')^{-1} | - h \left( (\mathbf{e}')' \mathbf{1}_{N \times 1} \right) \right\}.
\]

Recognizing that \( \Sigma_j (\mathbf{e}_j)^{-1} = M \cdot Id_N + \text{diag} (\mathbf{e}) \), and invoking results of the matrix calculus, the FOC for the optimal level of effort \( \mathbf{e}_i \) is:

\[
\text{Tr} \left[ \left( \Omega^{-1} + M \cdot Id_N + \text{diag} (\mathbf{e}) \right)^{-1} J_i \right] - h' (\mathbf{e}' \mathbf{1}_{N \times 1}) \leq 0 \quad (= \text{if } e_i > 0).
\]

where \( J_i \) is the \( N \times N \) matrix with entry \( J_{ii} = 1 \) and zero otherwise. Since \( \text{Tr} \) is a linear
operator, we can stack all the FOCs to arrive at:

$$Diag \left[ (\Omega^{-1} + M \cdot Id_N + diag(\mathbf{e}))^{-1} \right] - h'(\mathbf{e}'1_{N \times 1})1_{N \times 1} \leq 0_{N \times 1},$$

where $Diag$ is the operator that stacks the diagonal of a matrix into a vector. Furthermore, the second-order derivative of $\log |\Omega^{-1} + \Sigma_j (\mathbf{e}')^{-1}|$ is:

$$\partial^2_{\mathbf{e}, \mathbf{e}} \log |\Omega^{-1} + \Sigma_j (\mathbf{e}')^{-1}| = - (\Omega^{-1} + M \cdot Id_N + diag(\mathbf{e}))^{-1} J_i (\Omega^{-1} + M \cdot Id_N + diag(\mathbf{e}))^{-1}.$$

Since $h'(\cdot)$ is a (weakly) convex function, the optimization program is concave in $\mathbf{e}$, and therefore the FOC is both necessary and sufficient for the optimal $\mathbf{e}$.

If $F$ is diagonal, so that asset payoffs are independent, then $\Omega^{-1}$ is also diagonal, and the above condition reduces to:

$$\frac{1}{\Omega^{-1}_{ii} + M + \mathbf{e}_i} \leq h'(\mathbf{e}'1_{N \times 1}) \forall i \in \{1, ..., N\}.$$

When instead the participation constraint binds, then effort is chosen such that the level of active manager utility is $u_0$, or:

$$\frac{1}{2} h ((\mathbf{e}')'1_{N \times 1}) - \frac{1}{2} \log |Id_N + \Sigma_j (\mathbf{e}')^{-1} \Omega| = \log (-u_0) + \gamma_M \rho_0$$

$$- \log E \left[ \exp \left( -\frac{1}{2} (\hat{\Theta} - R^f F^{-1} \mathbf{P})' \Omega^{-1} (\hat{\Theta} - R^f F^{-1} \mathbf{P}) \right) \right]$$

(A5)

**A.4 Proof of Proposition 2**

Substituting for $W_2^I$ and $C_0^S$, the utility of investors is:

$$V(W_2^I, C_0^S) = -\exp \left( -\gamma \left( \left( R^f W_0 - y_i \rho_0 + y_i \frac{1 - \rho_S}{\gamma_M \rho_S} (s_j - \hat{\Theta})' \Sigma_j (\mathbf{e}_j)^{-1} (\Theta - R^f F^{-1} \mathbf{P}) \right) + \left( 1 - y_i \right) \rho_S \gamma \left( \hat{\Theta} - R^f F^{-1} \mathbf{P} \right)' \Omega^{-1} (\hat{\Theta} - R^f F^{-1} \mathbf{P}) \right) \right),$$

Importantly, $\mathbf{e}_j$ is independent of the realization of $\Theta$. To find expected investor utility when investing with fund managers, we recognize by the law of iterated expectations that $E \left[ V(W_2^I, C_0^S) | \mathcal{F} \right] = E \left[ E \left[ V(W_2^I, C_0^S) | \mathcal{F} \right] | \Theta, x \right]$, and that $E \left[ V(W_2^I, C_0^S) | \mathcal{F} \right] = E \left[ E \left[ V(W_2^I, C_0^S) | \Theta, x \right] | \mathcal{F} \right]$. Taking conditional expectations with respect to the realized shocks, and integrating over the
idiosyncratic signal noise of fund managers, we find:

\[
E \left[ V \left( W^*_1, C_0^S \right) | \Theta, x \right] = - \exp \left( - \gamma \left( R^f W_0 - y_i \rho_0 + y_i \frac{1-\rho_S}{\gamma_S} \left( 1 - y_i \frac{1-\rho_S}{\gamma_S} \right) \left( \Theta - \hat{\Theta} \right) \left( \Theta - R^f F^{-1} P \right) \right) \right),
\]

where \( Z = \hat{\Theta} - R^f F^{-1} P \). Taking conditional expectations with respect to the market beliefs, we then arrive at:

\[
E \left[ V \left( W^*_1, C_0^S \right) | F^c \right] = - \exp \left( \frac{\gamma y_i \rho_0 - \gamma R^f W_0 - \frac{1}{2} \left( 1 - y_i \frac{1-\rho_S}{\gamma_S} \right) \left( \Theta - \hat{\Theta} \right) \left( \Theta - R^f F^{-1} P \right) }{\left| \Omega_Z^{1/2} I_{d_N} + 2 y_i \frac{1-\rho_S}{\gamma_S} \left( 1 - y_i \frac{1-\rho_S}{\gamma_S} \right) \Omega \Sigma_j (e_j)^{-1} \right|^{1/2}} \right),
\]

From an ex ante perspective, \( Z \sim \mathcal{N} (\mu, \Omega_Z) \). Taking unconditional expectations, we arrive at:

\[
E \left[ V \left( W^*_1, C_0^S \right) \right] = - \exp \left( \frac{\gamma y_i \rho_0 - \gamma R^f W_0 + \frac{1}{2} \left( y_i \frac{1-\rho_S}{\gamma_S} \right)^2 \rho R' F \left( \theta - \hat{\Theta} \right) \\ \left( 1 + 2 y_i \frac{1-\rho_S}{\gamma_S} \left( 1 - y_i \frac{1-\rho_S}{\gamma_S} \right) \Omega \Sigma_j (e_j)^{-1} \right)^{-1} F R \right) \right),
\]

where

\[
Q = \Omega_Z^{-1} + \Omega^{-1} - y_i^2 \left( 1 - \frac{1-\rho_S}{\gamma_S} \right)^2 \Omega^{-1} \left( 1 + 2 y_i \frac{1-\rho_S}{\gamma_S} \left( 1 - y_i \frac{1-\rho_S}{\gamma_S} \right) \Omega \Sigma_j (e_j)^{-1} \right)^{-1} \Omega^{-1},
\]

\[
G = y_i^2 \frac{1-\rho_S}{\gamma_S} \Omega^{-1} \left( 1 + 2 y_i \frac{1-\rho_S}{\gamma_S} \left( 1 - y_i \frac{1-\rho_S}{\gamma_S} \right) \Omega \Sigma_j (e_j)^{-1} \right)^{-1} F R + \Omega_Z^{-1} \mu.
\]
Investors in fund managers are used to solve the optimization problem:

\[
V_0 = \sup_{\rho_0, \rho_S, \rho_R} E \left[ V \left( W_2^I, C_0^S \right) \right]
\]

s.t. : \( \text{Diag} \left[ \left( \Omega^{-1} + M \cdot Id_N + \text{diag} \left( e_j \right) \right)^{-1} J_i \right] - h^j \left( e_j^1 1_{N \times 1} \right) \leq 0 \ \forall \ i \in \{1, ..., N\} \) (optimal \( e_j \)).

Importantly, the FOC for the optimal choice of active manager effort is independent of the contract from the perspective of investors.

The FOC for \( \rho_R \) is:

\[
0 = \frac{\gamma}{\rho_S} \rho_R F' \Omega Q + \left( 1 - \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \mu' \Omega_1^{-1} \\
+ \gamma_2 \left( 1 - \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right)^2 \rho_R F' \left( \Omega^{-1} + 2 y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \left( 1 - \frac{y_i}{2 \gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \Sigma_j \left( e_j \right)^{-1} \right)^{-1} \Omega^{-1}
\]

which, with some manipulation, gives:

\[
\rho_R = - \left( \frac{\rho_S}{\gamma} - \frac{1 - \rho_S}{\gamma_M} \right) F^{-1} \left( \Omega_Z + \Omega \right)^{-1} \mu.
\]

Recognizing that we can rewrite \( F^{-1} \left( \Omega_Z + \Omega \right)^{-1} \mu \) as \( (F' \left( \Omega_Z + \Omega \right) F)^{-1} F \mu \), where \( F \mu \) is the unconditional expected excess return on the risky assets, it follows that \( F^{-1} \left( \Omega_Z + \Omega \right)^{-1} \mu \) is a portfolio allocation chosen before prices are observed that accounts for both the overall uncertainty of excess returns and the uncertainty given common prices \( \Omega \) augmented by the uncertainty over the realization of prices \( \Omega_Z \).

Defining \( \omega^0 = \frac{1}{\gamma} F^{-1} \left( \Omega_Z + \Omega \right)^{-1} \mu \) to represent this “naive” portfolio, we can express \( \rho_R \) as:

\[
\rho_R = - \left( \rho_S - \frac{\gamma}{\gamma_M} \left( 1 - \rho_S \right) \right) \omega^0.
\]

Furthermore, by the law of total variance:

\[
\text{Var} \left( \Theta - R^j F^{-1} P \right) = \text{E} \left[ \text{Var} \left( \Theta - R^j F^{-1} P \mid F^c \right) \right] + \text{Var} \left( \text{E} \left[ \Theta - R^j F^{-1} P \mid F^c \right] \right) = \text{E} \left[ \Omega \right] + \text{Var} \left( \Theta - R^j F^{-1} P \right) = \Omega + \Omega_Z.
\]

Therefore, \( \omega^0 \) can be expressed as:

\[
\omega^0 = \frac{1}{\gamma} F^{-1} \text{Var} \left( \Theta - R^j F^{-1} P \right)^{-1} \text{E} \left[ \Theta - R^j F^{-1} P \right],
\]

48
which is the ex-ante mean-variance efficient portfolio.

The optimal choices of $\rho_S$ satisfies the FONC:

$$\rho_S = \arg \sup_{\rho_S} \left\{ \frac{1}{2} \left( y_i \frac{\gamma}{\rho_S} \right)^2 \rho_R' F \left[ \Omega^{-1} + 2 y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \left( 1 - y_i \frac{1}{2} \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \Sigma_j (e_j)^{-1} \right]^{-1} F' \rho_R \right. \\
+ \frac{1}{2} \log |I_d N + 2 y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \left( 1 - y_i \frac{1}{2} \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \Omega \Sigma_j (e_j)^{-1}| + \frac{1}{2} \log |Q| - \frac{1}{2} \gamma' Q^{-1} G \right\}$$

Applying matrix calculus, we derive the FONC for the optimal choice of $\rho_S$:

$$0 = \left\{ \frac{1}{\rho_S} \left( y_i \frac{\gamma}{\rho_S} \right)^2 \rho_R' F A F' \rho_R \right. \\
- y_i \frac{\gamma}{\gamma_M} \left( y_i \frac{\gamma}{\rho_S} \right)^2 \left( 1 - y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \rho_R' F A \left[ \frac{1}{\gamma_M} \left( 1 - y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \Omega \Sigma_j (e_j)^{-1} \right] \} \\
- \frac{1}{\rho_S} \left( y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \rho_R' F A \left[ \frac{1}{\gamma_M} \left( 1 - y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \Omega \Sigma_j (e_j)^{-1} \right] \} \\
- \frac{1}{\rho_S} \left( y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \rho_R' F A \left[ \frac{1}{\gamma_M} \left( 1 - y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \Omega \Sigma_j (e_j)^{-1} \right] \} \\
y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \left[ \left( 1 - y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \Omega^{-1} \right] \{ I_d N - 2 y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \left( 1 - y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \Omega^{-1} \Sigma_j (e_j)^{-1} \}
$$

where:

$$A = \left( \Omega^{-1} + 2 y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \left( 1 - y_i \frac{\gamma}{\gamma_M} \frac{1 - \rho_S}{\rho_S} \right) \Sigma_j (e_j)^{-1} \right)^{-1}.$$ 

Suppose that we conjecture that $\rho_S = \frac{\gamma}{(\gamma + \gamma_M)}$, its value under perfect risk-sharing. Then $\rho_R = 0$, and substituting this into the FOC (A.4), we find that:

$$\frac{d \log (-V_0)}{d \rho_S} = \left\{ -y_i \frac{(\gamma + \gamma_M)^2}{\gamma M} \left( 1 - y_i \right) \text{Tr} \left[ \left( I_d N + 2 y_i \left( 1 - \frac{1}{2} y_i \right) \Omega \Sigma_j (e_j)^{-1} \right)^{-1} \Omega \Sigma_j (e_j)^{-1} \right] \right\} < 0.$$ 

(A6)

Assuming the program for the investor is concave in $\rho_S$, it then follows that $\rho_S \leq \frac{\gamma}{\gamma + \gamma_M}$. 

49
Assuming the program of investors is concave in $y_i$, the FOC for $y_i$ is given by:

\[
\rho_0 = -\frac{1}{\gamma} y_i \frac{1}{\rho_S} \rho R F \left[ \Omega^{-1} + 2 y_i \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S} \left( 1 - y_i \frac{1}{2} \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S} \right) \Sigma_j (e_j)^{-1} \right]^{-1} F' \rho R + \frac{1}{\gamma} \frac{1-\rho_S}{\rho_S} \left( y_i \frac{1}{\gamma_M} \frac{1-\rho_S}{\rho_S} \right) y_i \left( \frac{\gamma}{\rho_S} \right)^2 \text{Tr} \left[ A F' \rho R \rho R F A \Sigma_j (e_j)^{-1} \right] + \text{Tr} \left[ A \Sigma_j (e_j)^{-1} \right]
\]

\[
+ \frac{1}{\gamma} y_i \left( 1 - \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S} \right)^2 \text{Tr} \left[ H^{-1} A \Omega \Sigma_j (e_j)^{-1} \Omega^{-1} \left( y_i \frac{1}{\gamma_M} \frac{1-\rho_S}{\rho_S} \right) \Omega \Sigma_j (e_j)^{-1} \Omega^{-1} - \text{Id}_N \right]
\]

\[
+ \frac{1}{\gamma} y_i \left( 1 - \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S} \right)^2 \text{Tr} \left[ H^{-1} G G' H^{-1} A \Omega \Sigma_j (e_j)^{-1} \Omega^{-1} \left( y_i \frac{1}{\gamma_M} \frac{1-\rho_S}{\rho_S} \right) \Omega \Sigma_j (e_j)^{-1} \Omega^{-1} - \text{Id}_N \right]
\]

\[
+ y_i \frac{1}{\rho_S} \left( 1 - \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S} \right) G' H^{-1} \left[ y_i \frac{1}{\gamma_M} \frac{1-\rho_S}{\rho_S} \left( 1 - y_i \frac{1}{\gamma_M} \frac{1-\rho_S}{\rho_S} \right) \Omega^{-1} A \Sigma_j (e_j)^{-1} \Omega - \text{Id}_N \right] \Omega^{-1} A F' \rho R
\]

Since $y_i = \chi$ in equilibrium, this pins down $\rho_0$.

### A.5 Proof of Proposition 3

When the participation constraint does not bind, define:

\[
G = \text{Tr} \left[ X^{-1} J_i \right] - h' (e' 1_{N \times 1}) = 0,
\]

where: $X = \left( (\tau_0 + M) \cdot \text{Id}_N + k (M \cdot \text{Id}_N + \text{diag}(e)) (F'F)^{-1} (M \cdot \text{Id}_N + \text{diag}(e)) + \text{diag}(e) \right) J_{ii}$, and $k = \tau_z \left( \frac{\chi}{\gamma_M \rho_S} \right)^2$. By the implicit function theorem:

\[
\partial_z e_i = -\frac{\partial_z G}{\partial e_i G},
\]

for parameter $z$. Recognizing that $\partial (X^{-1}) = X^{-1} (\partial X) X^{-1}$, taking the derivative under the $\text{Tr}$ operator since the $\text{Tr}$ operator is linear and the trace is bounded, it follows that:

\[
\partial_e G = \text{Tr} [AJ_i] = v'_i A v_i - h'' (e' 1_{N \times 1})
\]

where $J_i$ is the $N \times N$ matrix with entry $J_{ii} = 1$ and zero otherwise, $v_i$ is the Euclidian $N \times N$ basis vector in the $i^{th}$ direction, and:

\[
A = -X^{-1} \left( k (M \cdot \text{Id}_N + \text{diag}(e)) F^{-1} F' + k (F'F)^{-1} (M \cdot \text{Id}_N + \text{diag}(e)) + \text{Id}_N \right) X^{-1}
\]

\[
- h'' (e' 1_{N \times 1}) \text{Id}_N.
\]

Given that $F$ is a lower triangular matrix with entries of 1 on the diagonal, $F'F$ is a positive definite (PD) matrix since $\det(AB) = \det(A) \det(B)$. Since $F'F$ is a positive definite (PD) matrix, it follows that $(F'F)^{-1}$ is a PD matrix, since the eigenvalues of $(F'F)$ are the inverse of the eigenvalues of
Since $(F'F)^{-1}$ is a PD matrix, $X$ is also a PD matrix, and it follows that $A$ is a negative definite (ND) matrix. Therefore, it follows since $v_i$ has non-negative entries and $h(e'1_{N	imes1})$ is convex that:

$$\partial e_i G = v_i' A v_i - h''(e'1_{N	imes1}) < 0.$$

Consequently:

$$\partial_z e_i = \frac{\partial_z G}{|\partial e_i G|} = \frac{\partial_z Tr [X^{-1}J_i]}{|\partial e_i G|},$$

and the sign of $\frac{\partial e_i}{\partial z}$ is the same as the sign of $\partial_z Tr [X^{-1}J_i]$. Differentiating under the $Tr$ operator again, it follows that:

$$\partial_z G = -Tr \left[ X^{-1} \partial_z \left( (\tau_0 + M) \cdot Id_N + k (M \cdot Id_N + \text{diag}(e)) (F'F)^{-1} (M \cdot Id_N + \text{diag}(e)) \right) X^{-1}J_i \right].$$

For $z = k$, it is then follows that $\partial_k G > 0$, and therefore:

$$\partial_k e_i < 0.$$

The result for $\rho_S$ then follow by the chain rule for $k$. As this $\rho_S$ is the industry $\rho_S$, we refer to it as $\hat{\rho}_S$ in the proposition.