Expected Utility and Risk on Stochastic Social Influence Networks

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Abstract

Expected utility and risk analysis for stochastic influence networks requires accounting for the complex interconnections between agents as well as between agents and stochastic phenomena. Conditional game theory provides a framework within which to study expected utility and risk for such networks. Key features of this approach are a) mechanism for modulating individual preferences in response to social and stochastic influence; b) a mechanism for aggregating social and stochastic influence to generate a comprehensive stochastic social model; c) an operational definition of coordination as a measure of the shared intentionality of a network; d) analysis of expected utility and risk for each member of the network; and e) analysis of the expected coordination utility and risk of the network.

1 Introduction

A social influence network is a collective of volitional entities (agents) who are connected by some means of communication or control that enables them to respond to the influence exerted on each other. A social network is stochastic if it includes stochastic entities (random states of nature) with behavior governed by probability distributions. Such stochastic entities may influence the values of the volitional entities as well as other stochastic entities, and volitional entities may influence stochastic entities as well as other volitional entities. As influence diffuses throughout the network, the result is an emergent social conglomeration that may exhibit some form of coordinated behavior (which could range from anarchy to a well-designed organization). Coordination may be cooperative, such as the formation of a business partnership, it could be conflictive, such as a military engagement between enemies; it could be mixed, such a family, where some members are cooperative and some are conflictive; or it could fail to correspond to any discernible notion of systematic behavior. An adequate model of such a system must account for the effects of both social and stochastic influence, and must enable an assessment of the risk associated with behavior in terms of both individual level performance and group level coordination. To illustrate the structure of such a network, consider the network

\[ \xymatrix{ & X_1 \\
Y & X_2 \ar[ul] & X_1 
} \]

where the vertices \( X_1 \) and \( X_2 \) are volitional entities (agents) and the vertex \( Y \) is a stochastic entity (a random variable). The directed edges express influence propagation, with the link \( X_1 \rightarrow Y \) expressing the influence that \( X_1 \) exerts on \( Y \) and the link \( \{X_1, Y\} \implies X_2 \) expressing the influence that \( X_1 \) and \( Y \) jointly exert on \( X_2 \).
The goal of this study is to a) define a diffusion mechanism for the propagation of influence throughout the network and the expression of individual preferences as modulated by social/stochastic influence; b) define an aggregation mechanism for the generation of emergent social/stochastic interrelationships and the development of socially rational solution concepts; and c) generate concepts of expected utility and risk that account for all social/stochastic interrelationships.

Social psychologists and mathematicians have studied social influence network theory since the 1950s, with much of the research focusing on the organizational structure of so-called small groups, defined as loosely coupled collectives of mutually interacting individuals (Weick, 1995). Specifically, much of the emphasis has been placed on the structure of such organizations (French, 1956; Friedkin, 1986; Friedkin and Johnson, 1997; Friedkin, 1998; Friedkin and Johnson, 2011). The basic model is that an individual’s socially adjusted preference is a convex combination of its own payoff and a weighted sum of the payoffs of those agents who influence it. Game theory has been widely used as a framework within which to study coordination (cf. Schelling, 1960; Lewis, 1969; Bicchieri, 1993; Cooper, 1999; Hoede and Bakker, 1982; Hu and Shapley, 2003; Bacharach, 2006; Nisan et al., 2007; Goyal, 2007; Jackson, 2008; Shoham and Leyton-Brown, 2009; Grabisch and Rusinowska, 2010; Easley and Kleinberg, 2010; Sugden, 2014). All of these approaches, however, share a common feature: they separate the way preferences are specified from the way solutions are defined. They follow the dictum expressed by Friedman that “economic theory proceeds largely to take wants as fixed. This is primarily a case of division of labor. The economist has little to say about the formation of wants; this is the province of the psychologist. The economist’s task is to trace the consequences of any given set of wants” (Friedman, 1962, p. 13). The conventional way agents express their preferences is with payoff functions that are fixed, immutable, and unconditional—they are categorical. Each agent must incorporate all material and social considerations regarding each outcome into a single categorical preference ordering. These payoffs are then juxtaposed in a payoff array, to which a solution concept, such as Nash equilibrium, can be applied.

The assumption that one’s preferences can be adequately expressed by a single ordering has long been criticized. As Sen observed, “The purely economic man is indeed close to being a social moron. Economic theory has been much preoccupied with this rational fool decked in the glory of his one all-purpose ordering. To make room for the different concepts related to his behavior we need a more elaborate structure” (Sen, 1977, pp. 335 - 336). A single preference ordering may be an appropriate mechanism for one whose concept of rational behavior is limited to concern for, and only for, one’s personal benefit, but for environments involving influence (both social and stochastic), such a narrow focus may prove to be inadequate.

An essential attribute of an agent is its principle of rationality: the logic that motivates it to define and pursue its goals (Newell, 1982; Popper, 1994). The rationality principle that is often invoked is narrow self-interest. Descriptively, narrow self-interest means a focus on one’s own welfare without explicit consideration of the welfare of others. In terms of decision making, however, descriptive definitions are inherently too imprecise to define a principle of rationality. What is required is an operational definition in terms of quantitative attributes. The operational concept of narrow self-interest that is almost universally applied is a categorical linear preference ordering over the set of outcomes.

Although narrow self-interest could motivate the members of a conglomeration to maximize their expected individual material benefit, it may be more natural and important for them to respond in accordance with the nascent social structure that emerges as the individuals interrelate. It is more important for a business partnership to settle on a productive division of labor than for
each partner to maximize individual control. It is more important to win a war than simply to destroy as many enemy resources as possible. In fact, as Arrow has observed, reliance on a narrow concept of self-interest as a principle of rationality has a specific and limited application.

Rationality in application is not merely a property of the individual. Its useful and powerful implications derive from the conjunction of individual rationality and other basic concepts of neoclassical theory—equilibrium, competition, and completeness of markets. ... When these assumptions fail, the very concept of rationality becomes threatened, because perceptions of others and, in particular, their rationality become part of one's own rationality (Arrow, 1986, p. 203).

Conditional game theory, as developed and discussed by (Stirling, 2012; Stirling and Felin, 2013; Stirling, 2016), is an extension of conventional noncooperative game theory that is designed to accommodate complex models of social behavior, including models involving the diffusion of social and stochastic influence and the resulting impact on agent behavior as the agents interact. The key features of this approach are a) a mechanism to enable the diffusion of social and stochastic influence throughout the network and b) a mechanism to aggregate the emergent social relationships to create a comprehensive stochastic social model that enables the analysis of risk for the network.

2 Socially Rational Preference Concepts

2.1 Congruence

A game scenario comprises the following elements:

- a set $X = \{X_1, \ldots, X_n\}$ of $n \geq 2$ agents, each of which possesses a finite action set $A_i = \{x_{i1}, \ldots, x_{i,N_i}\}$;
- an outcome set $A = A_1 \times \cdots \times A_n$, comprising all possible combinations of actions or profiles of the form $a = (a_1, \ldots, a_n)$ where $a_i \in A_i$, $i = 1, \ldots, n$;
- a preference concept for each $X_i$ that defines the criteria and mechanism for its evaluation of the elements of $A_i$; and
- a solution concept for each $X_i$ that defines the criteria and mechanism for evaluating the elements of its action set $A_i$.

The distinction between a preference concept and a solution concept is critical. The former involves the evaluation of group-level outcomes, and the latter involves the evaluation of individual-level actions. This is the inescapable dichotomy of game theory: how to reconcile preferences defined over group-level outcomes with solutions defined over individual-level actions. For the game to be well-formed, it is essential that the preference and solution concepts must be compatible. A game is rationally congruent if the preference and solution concepts are governed by the same principle of rationality.

Since the preferences for a conventional noncooperative game are expressed by categorical payoffs, the governing rationality principle is narrow self-interest. For the game to be rationally congruent, the solution concept must also be governed by narrow self-interest, which is indeed the case for solution concepts such as Nash equilibria, since each agent seeks to maximize its payoff under the constraint that others are likewise motivated. However, if the solution concept invokes notions of sociality that go beyond narrow self-interest, then congruence will be violated. An example is the
Prisoner’s Dilemma. Mutual defection is the unique Nash equilibrium, and invoking that solution concept ensures that the game is congruent. Empirical studies reveal, however, that people often choose to cooperate, which implies that they are invoking a de facto principle of rationality that is different from narrow self-interest, thereby violating congruence.

A principle of rationality that fits well with the concept of individuals functioning in their emergent role as part of a conglomeration is the notion of coordination. As defined by the Oxford English Dictionary, to coordinate is “to place or arrange (things) in proper position relative to each other and to the system of which they form parts; to bring into proper combined order as parts of a whole” Murray et al. (1991). This suggests that agents (the parts) may extend their spheres of interest to fit the social environment defined by the collective (the whole) in which they exist without violating their own intrinsic individuality. By so doing, they are replacing narrow self-interest with what might be termed coordinated self-interest: the concept of being willing and able to modulate their preferences in response to their social environment in the pursuit of their individual goals while retaining their own innate volition. Coordinated self-interest is related to the notion of “enlightened self-interest” (Dwan and Insole, 2012): the doctrine that individuals better serve their own goals by enabling others to serve their goals. The difference between the two notions is that enlightened self-interest has a definite positive connotation, while coordinated self-interest has a neutral connotation that applies to both cooperative and competitive scenarios.

The concept of coordinated self-interest comports with Arrow’s assertion that the one’s rationality is influenced by the rationality of others. As it stands, however, the concept is too imprecise to be operational. Thus, just as categorical payoffs serve as the operational concept of narrow self-interest, it is necessary to establish a mathematically precise operational definition of coordinated self-interest that can be used to define socially rational preference and solution concepts. To do so requires a mathematical model to characterize the way influence is diffused throughout a network.

2.2 Conditionalization

The mechanism used by conditional game theory to diffuse influence is conditionalization—the concept of defining preferences in terms of hypothetical propositions. The concept is relatively straightforward, but developing it requires a multitude of definitions. Once established, however, the concept is intuitive and accessible. In the interest of clarity, this concept is developed first for non-stochastic networks.

An influence network is expressed as a directed graph whose vertices are agents and whose edges define the influence relationships between parents and children. If $X_j$ influences $X_i$, then there is a directed edge, denoted $X_j \rightarrow X_i$, from $X_j$ to $X_i$.

A conjecture profile $a_i = (a_{i1}, \ldots, a_{in}) \in A$ by $X_i$ is a hypothetical assertion, denoted $X_i \models a_i$, that $X_i$ intends $a_i$ to be actualized. A self-conjecture, denoted $X_i \models a_{ii}$, is a hypothetical assertion that $X_i$ intends to actualize $a_{ii}$. A joint conjecture profile, denoted $\alpha_{i:n} = (a_1, \ldots, a_n)$, is a set of conjecture profiles for the collective.

The parent set for $X_i$, denoted $pa(X_i) = \{X_{i1}, \ldots, X_{ip_i}\}$, is the subset of agents such that $X_{ij} \rightarrow X_i$, $j = 1, \ldots, p_i$. If $pa(X_i) = \emptyset$, then $X_i$ is a root vertex. A conditioning conjecture profile by $X_i$ for $X_{ij}$ is a conjecture profile $a_{ij} = (a_{ij1}, \ldots, a_{ijn}) \in A$ that $X_i$ hypothesizes that $X_{ij}$ intends to be actualized, denoted $X_{ij} \models a_{ij}$. A conditioning conjecture set $\alpha_{pa(i)} = (a_{i1}, \ldots, a_{ip_i})$ is the set of conditioning conjecture profiles by $X_i$ for $pa(X_i)$, denoted $pa(X_i) \models \alpha_{pa(i)}$.

A conjecture hypothetical proposition, denoted

$$H(a_i|\alpha_{pa(i)}): \ pa(X_i) \models \alpha_{pa(i)} \Rightarrow X_i \models a_i,$$  \hfill (2)
is a hypothetical proposition with antecedent \( \text{pa} (X_i) = \alpha_{\text{pa}(i)} \) and consequent \( X_i = a_i \); that is, if \( \text{pa} (X_i) \) conjectures \( \alpha_{\text{pa}(i)} \), then \( X_i \) conjectures \( a_i \).

The function \( v_{i|\text{pa}(i)}(a_i|\alpha_{\text{pa}(i)}) \) is a conditional payoff for \( H(a_i|\alpha_{\text{pa}(i)}) \), and defines the reward to \( X_i \) if the outcome \( a_i \) is actualized, given that \( X_i \) conjectures that outcome \( a_i \) should or will be actualized, \( j = 1, \ldots, p_i \).

For many applications, the full generality of conditional payoff is not required. An important class of interactive decision problems comprises those for which the agents are more immediately concerned regarding the self-conjectures of its parents, rather than their conjectured outcomes. In such cases, the conditioning conjecture profiles \( a_{ij} \) may be replaced by conditioning self-conjectures, denoted \( a_{ij} \in A_{ij}, j = 1, \ldots, p_i \), and the conditioning conjecture set collapses to become \( \alpha_{\text{pa}(i)} = (a_{i1}, \ldots, a_{ip_i}) \). Such an agent is said to be conjecture dissociated.

Another important class of interactive decision problems comprises those for which the agents are more immediately concerned with their actions rather than the outcome, given the influence exerted on them. In such cases, the player’s conjecture profile \( a_i \) may be replaced by its self-conjecture. Such an agent is payoff dissociated. A player who is both conjecture dissociated and payoff dissociated is said to be dissociated. If the agent is neither conjecture dissociated nor payoff dissociated, it is sociated. All other cases are partially sociated. To illustrate, consider the two-agent network

\[
\begin{array}{c}
X_1 \\
\uparrow_{v_{2|1}} \\
X_2
\end{array}
\]  

where \( X_1 \) possesses a categorical payoff \( v_1 \) and \( X_2 \) possesses a conditional payoff \( v_{2|1} \). If the players are sociated, then the payoffs are of the form \( v_1(a_{11}, a_{12}) \) and \( v_{2|1}(a_{21}, a_{22}|a_{11}, a_{12}) \), and if they are dissociated, then the payoffs are of the form \( v_1(a_{11}) \) and \( v_{2|1}(a_{22}|a_{11}) \). Since these notions of dissociation are special cases of sociation where the full conjecture profiles are replaced by self-conjectures, only the general theory need be presented.

**Definition 1.** A conditional game is a triple \((X, A, V)\) with the players set: \( X = \{X_1, \ldots, X_n\} \), the outcome set: \( A = A_1 \times \cdots \times A_n \), and conditional payoff set:

\[
V = \{v_{i|\text{pa}(i)}(\cdot|\alpha_{\text{pa}(i)}) \mid \forall \alpha_{\text{pa}(i)} \in A^{p_i}, i = 1, \ldots, n\}
\]  

A conditional game degenerates to a conventional noncooperative normal-form game when \( V = \{v_i, i = 1, \ldots, n\} \). Thus, conditional game theory is an an extension of conventional noncooperative game theory.

Conditional payoffs enable agents to modulate their preferences in response to the social influence exerted by their parents. This mechanism allows agents to expand their individual rationality by incorporating the interests of others into their own rationality without surrendering their own intrinsic individual volition.

### 3 Socially Rational Solution Concepts

#### 3.1 Aggregation

The existence of social influence relationships among the members of a network invites the development of notions of group-level behavior. The concept of a “group value” or “group payoff,” however, is generally eschewed by noncooperative game theory. Shubik (1982) warns against the “anthropomorphic trap” of ascribing preference to a group: “It may be meaningful, in a given setting, to say that group ‘chooses’ or ‘decides’ something. It is rather less likely to be meaningful
to say that the group ‘wants’ or ‘prefers’ something” (p. 124). Ascribing material value to a group is inherently problematic, since game theory requires that preferences be defined by individuals. Nevertheless, as the members of a group interrelate, a notion of shared intentionality may emerge. Bratman (2014) introduces the concept of “augmented individualism,” which asserts that there is no discontinuity between individual and joint intentionality: “shared intention consists primarily of interrelated attitudes (especially intentions) . . . and that the contents of the attitudes that are constitutive of basic cases of shared intention need not in general essentially involve the very idea of shared intention (though on occasion they may)” (p. 12). According to this thesis, shared intentionality is an emergent phenomenon that arises endogenously as the agents interact.

In the context of social influence networks, shared intentionality is compatible with the notion of coordination used herein.

In terms of network functionality, it is often the case that the propensity of a group to coordinate is more relevant than the propensity of the individuals to optimize. Focusing on individual performance without considering group coordination is an incomplete characterization of group behavior. Similarly, focusing on group coordination without considering individual performance is an incomplete characterization of individual behavior. Coordination without performance is unproductive, and performance without coordination is equivocal. A full understanding of the functionality of a group requires the assessment of both attributes.

As the conditional preferences diffuse throughout the network, nascent social interrelationships are established between its members. The aggregation of these interrelationships creates a comprehensive social model of the network that incorporates all social influence. Given a conditional game \( \{ X, A, V \} \), an aggregation functional as a mapping \( F: V \rightarrow [0,1] \) that generates a social model \( v_{1:n}: A^n \rightarrow [0,1] \) of the form

\[
v_{1:n}(a_1, \ldots, a_n) = F[v_i|\text{pa}(i)(a_i|\alpha_{\text{pa}(i)}), i = 1, \ldots, n].
\]

(5)

To ensure that aggregation is well defined, it is necessary to require that pathological situations that threaten the autonomy and independence cannot occur. In particular, it is essential that no agent be subjugated. To define this condition, let \( a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in A^{n-1} \) denote the subset of the joint conjecture profile that excludes \( a_i \). \( X_i \) is subjugated if, for any \( a \in A \),

\[
v_i(a) > v_i(a') \forall a' \neq a,
\]

(6)

then

\[
v_{1:n}(a, a_{-i}) < v_{1:n}(a', a_{-i}) \forall a' \neq a, \quad \forall a_{-i} \in A^{n-1}.
\]

(7)

If \( X_i \) is subjugated, then, no matter what conjecture profile it most prefers, all joint conjecture profiles that include its most preferred conjecture profile are dominated by all other joint conjecture profiles. In other words, the fact that \( X_i \) most prefers any outcome distorts the social model. Notice that if the inequality in (7) were reversed, then \( X_i \) would become a subverter—one who unilaterally controls the ordering of the aggregated behavior of the group. Both subjugation and subversion are pathological social attributes that produce dysfunctional behavior. Thus, a minimal concept of a democratic network is that subjugation and its dual, subversion, must be impossible.

The key result of this section is that necessary and sufficient conditions to avoid subjugation are isomorphic to necessary and sufficient conditions to avoid a sure loss—a gambling scenario, termed a Dutch book, such that no matter what the outcome of the wager, the entry fee will exceed the payoff.

**Theorem 1.** Subjugation is isomorphic to sure loss.
Proof. Let $\succeq_b$ and $\succeq_p$ denote linear belief and preference orderings over $\mathcal{A}$. Since both orderings are complete, there exists a permutation $\pi : \mathcal{A} \to \mathcal{A}$ such that, for any pair $(a, a') \in \mathcal{A} \times \mathcal{A}$, $a \succeq_b a'$ if, and only if, $\pi(a) \succeq_p \pi(a')$. Since $\pi$ is bijective, it is an order isomorphism.

Let $v_i$ denote a categorial payoff defined over the product action set $\mathcal{A}$. By the order isomorphism, $v_i$ may also be interpreted as a belief function. Also, the social model $v_{1:n}$ may be interpreted as a belief function defined over the product event set $\mathcal{A}^n$. Suppose a gambler enters a lottery to win $\$1$ if $a$ is realized. On the basis of (6), a fair entry fee is $\$p > 1/2$. Suppose, also, that on the basis of (7), the gambler enters a lottery to win $\$1$ if $a$ is not realized, with a fair entry fee of $\$q > 1/2$. The gambler is sure to win exactly $\$1$ by entering both lotteries, but a sure loss results, since the total entry fee is $\$(p + q) > 1$.

The Dutch book theorem and its converse establish that a sure loss is impossible if, and only if, the gambler’s beliefs are consistent with the axioms of probability theory. Thus, by the order isomorphism, subjugation is impossible if, and only if, payoffs are expressed and aggregated according to the syntax of probability theory. In accord with the term coherence as introduced by de Finetti (1990), a group is socially coherent if subjugation (and subversion) is impossible.

Since payoffs are unique up to positive affine transformations, it may be assumed without loss of generality that all payoffs are nonnegative and sum to unity and thus become mass functions with the same syntax as probability mass functions. Consequently, an acyclical conditional game network is isomorphic to a Bayesian network. The key difference between the two interpretations is that, with a Bayesian network, vertices are random variables with beliefs governed by probability mass functions, whereas the vertices of a condition game network are agents with preferences governed by conditional payoff mass functions.

The isomorphic relationship between a conditional game network and a Bayesian network makes it possible to apply a fundamental theorem of Bayesian networks, the Markov condition: nondescendent nonparents of a vertex have no influence on the vertex, given the state of its parent vertices (Cozman 2000). Consequently, the joint probability mass function of all vertices is uniquely given by the product of all conditional and categorical probability mass functions (Pearl 1988; Cowell et al. 1999; Jensen 2001). By the isomorphism, the unique socially coherent social model is the product of the categorical and conditional payoffs. This establishes the following result.

**Theorem 2.** For an acyclical conditional game to be socially coherent, the social model must be of the form

$$v_{1:n}(a_1, \ldots, a_n) = \prod_{i=1}^{n} v_{i} | pa(i)(a_i | \alpha_{pa(i)}) . \quad (8)$$

The social model is isomorphic to a joint probability mass Function of A set of random variables. Analogous to the way that a joint probability mass function serves as a comprehensive model of all statistical relationships among random variables, the social model serves as a comprehensive model of all social interrelationships among agents.

Although the social model may conceivably be used to define rational behavior from a group perspective, it is not immediately obvious how it may be used to define rational behavior from an individual perspective. It does, however, provide a social model from which individually rational solution concepts may be conceived.

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1To be precise, the vertices of a conditional game network are isomorphic to random vectors, but Bayesian network theory easily extends to that case.
3.2 Coordination

The ordering provided by the social model is with respect to joint conjecture profiles \( \alpha_{1:n} = (a_1, \ldots, a_n) \), with each conjecture profile of the form \( a_i = (a_{i1}, \ldots, a_{in}) \), where \( a_{ii} \in A_i \) is a conjecture by \( X_i \) and \( a_{ij} \in \mathcal{A}_j \) is a conjecture for \( X_j \) by \( X_i \). What is most relevant to coordination, however, is how the self-conjectures \( a_{ii}, i = 1, \ldots, n \) fit together to generate a coordinated outcome. Since each \( X_i \) has direct control over only \( a_{ii} \), its own self-conjecture, it is necessary to develop an expression that accounts for the way these self-conjectures combine to form a notion of coordination.

Given a joint conjecture profile \( \alpha_{1:n} = (a_1, \ldots, a_n) \), the coordination conjecture profile, denoted \( a = (a_{11}, \ldots, a_{nn}) \), comprises the \( i \)-th element of each \( X_i \)'s conjecture profile, \( i = 1, \ldots, n \). The coordination function, denoted \( w_{1:n} \), for \( \{X_1, \ldots, X_n\} \), is obtained by summing the social model over all elements of each \( a_i \) except the \( i \)-th elements, yielding

\[
w_{1:n}(a_{11}, \ldots, a_{nn}) = \sum_{-a_{11}} \cdots \sum_{-a_{nn}} v_{1:n}(a_{11}, \ldots, a_{1n}) \cdots (a_{n1}, \ldots, a_{nn}),
\]

where the notation \( \sum_{-a_{ii}} \) means that the sum is taken over all elements except \( a_{ii} \). Analogous to the way joint probability provides a comprehensive description of the interrelationships that exist among a set of discrete random variables as a result of stochastic interdependencies, the coordination function provides a comprehensive description of the interrelationships that exist among a set of agents as a result of social interdependencies. Unlike a group payoff, which would be a measure of material benefit to the group, coordination is a measure of group sociality that may or may not correspond to material benefit. A term that has recently come into vogue is the notion of social capital, as defined by Dictionary.com: “The interpersonal relationships, institutions, and other social assets of a society or group that can be used to gain advantage” (Dictionary.com, 2017). The Organization for Economic Cooperation and Development (OECD) also provides a definition: “networks together with shared norms, values and understandings that facilitate cooperation within or among groups” (OECD, 2017). Given these definitions, one interpretation of the coordination function is as a measure of social capital.

Also, analogous to the way the probability a discrete random variable is extracted form the joint probability of a collective random variables, The payoff for \( X_i \) is the \( i \)-th marginal of \( w_{1:n} \), that is,

\[
w_i(a_{ii}) = \sum_{-a_{ii}} w_{1:n}(a_{11}, \ldots, a_{1n}).
\]

As an example, consider the conditional game \( \{X_1, X_2\}, \mathcal{A}_1 \times \mathcal{A}_2, \{v_{X_1}, v_{X_2|X_1}\} \) with network

\[
\begin{array}{c}
X_1 \\
v_{2|1}
\end{array} \quad \begin{array}{c}
X_2
\end{array}
\]

The social model is

\[
v_{12}(a_{11}, a_{12}), (a_{21}, a_{22}) = v_1(a_{11}, a_{12})v_{2|1}(a_{21}, a_{22}|a_{11}, a_{12}),
\]

yielding the coordination

\[
w_{12}(a_{11}, a_{22}) = \sum_{a_{12}a_{21}} v_{12}(a_{11}, a_{12}), (a_{21}, a_{22})
\]

and payoffs

\[
w_i(a_{ii}) = \sum_{-a_{ii}} w_{12}(a_{11}, a_{22}), \ i = 1, 2.
\]
Coordination is a principle of behavior on a parallel with, but different from, performance. Performance deals with the efficiency and effectiveness of individual behavior in terms of material payoffs. Coordination, on the other hand, is an organizational property that characterizes the way the individuals fit together to form a coherent network. To put it succinctly, *individuals perform; groups coordinate.*

It is important to stress that $w_i$ serves as the operational measure of individual behavior in terms of performance, and $w_{1,n}$ serves as the operational measure of group behavior in terms of coordination. It may seem superfluous to offer different interpretations of these two measures. If payoffs were aggregated summatively (e.g., utilitarianism), then a notion of performance for the group may be meaningful—at least the payoffs would all be expressed in the same units. But the concept of aggregation used by conditional game theory is multiplicative, which raises the prospect of creating a concept of value different from the individual concepts. To illustrate, consider the network (11), and suppose $X_1$ represents electrical current and $X_2$ represents voltage. The product of current and voltage is electric power—an emergent phenomenon that is inherently “social” in the abstract sense in that it involves interrelationships between two distinct concepts. In a very real operational sense, electric power is the aggregation of current and voltage that produces a new phenomenon that could not be anticipated before their aggregation.

Coordination is an *emergent* property of a network. The players do not come to the social encounter with ex ante intentions of group-level behavior. Rather, any such group-level intentions emerge endogenously as their conditional preferences combine and thereby define a nascent notion of coordination. Coordination may be positive if the social influence is such that the agents engage in cooperation (e.g., teams). Coordination may be negative if the social influence is such that the agents engage in conflict (e.g., athletic contests and military engagements).

### 4 Networks with Stochastic States

#### 4.1 Stochastic Conditional Games

Since social coherence requires conditional payoffs to be defined and manipulated according to the syntax of probability theory, they are mathematically compatible with probability mass functions. This compatibility makes it possible to introduce stochastic phenomena into an influence network as parallel entities with volitional entities. Since both stochastic and volitional entities populate the network, it will be convenient to refer these entities as states, comprising stochastic states (random variables) and volitional states (agents).

Stochastic states can influence, and be influenced by, volitional states. For example, agents may be influenced by phenomena beyond their control, such as environmental factors, or by other agents who’s behavior is only known probabilistically. Consider the network

$$
\begin{array}{c}
Y_1 \\
\end{array} \xrightarrow{v_{X|Y_1}} X \xrightarrow{p_{Y_2|X}} Y_2,
$$

where $Y_1$ is a discrete random variable (a stochastic state) defined over a finite set $\mathcal{Y}_1$ that influences $X$, an agent (a volitional state) defined over an action set $\mathcal{A}$, and $Y_2$ is a discrete random variable (a stochastic state) defined over a finite set $\mathcal{Y}_2$ that is influenced by $X$. The function $v_{X|Y_1}$ has the syntax of a conditional payoff mass function whose conditioning entity is a stochastic state rather than another volitional state. Also, $p_{Y_2|X}$ has the syntax of a conditional probability mass function whose conditioning entity is a volitional state, rather than another stochastic state.

Given the isomorphism between belief and preference, coupled with the demand for coherence, the syntax of probability theory and the syntax of conditional payoff theory are identical. The fact...
that volitional states are deterministic decision makers and the stochastic states are determined by
nature is irrelevant to the mathematical structure of the network. The operation of payoff mass
functions and probability mass functions is the same as far as diffusing influence is concerned.

Consider a collective of $n$ volitional states, $X = \{X_1, \ldots, X_n\}$, each with a finite action set
$A_i$, and $m$ stochastic states $Y = \{Y_1, \ldots, Y_m\}$, where $Y_j$ is defined over a finite set $\mathcal{Y}_j$. Let
$\mathcal{A} = A_1 \times \cdots \times A_n$ denote the product actualization set for $\{X_1, \ldots, X_n\}$ and let $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m$
denote the product realization set for $\{Y_1, \ldots, Y_m\}$. The combined set $\{X, Y\}$ defined over the
product set $\mathcal{A} \times \mathcal{Y}$ comprises a stochastic network.

A stochastic conjecture profile set is an array $(a_1, \ldots, a_n, y_1, \ldots, y_m) \in \mathcal{A}^n \times \mathcal{Y}$ such that
$(a_1, \ldots, a_n)$ is a joint conjecture profile for $X$ and $(y_1, \ldots, y_m)$ is a stochastic profile for $Y$.

The presence of random phenomena in a network requires extending the concept of influence to
account for the social relationships that exist among the volitional states, the stochastic dependency
relationships that exist among the stochastic states, the stochastic relationships that the volitional
states exert on the stochastic states, and the social relationships that the stochastic states exert on
the volitional states.

Suppose $X_i$ has $p_i$ volitional parents and $q_i$ stochastic parents, and $Y_j$ has $r_j$ volitional parents
and $s_j$ stochastic parents, that is,

$$
\begin{align*}
\text{pa } (X_i) &= \{X_{i_1}, \ldots, X_{i_{p_i}}, Y_{i_1}, \ldots, Y_{i_{q_i}}\} \\
\text{pa } (Y_j) &= \{X_{j_1}, \ldots, X_{j_{r_j}}, Y_{j_1}, \ldots, Y_{j_{s_j}}\}.
\end{align*}
$$

(16)

For any stochastic joint conjecture profile $(a_1, \ldots, a_n, y_1, \ldots, y_m)$, let

$$
\alpha_{\text{pa}(i)} = (a_{i_1}, \ldots, a_{i_{p_i}}, y_{i_1}, \ldots, y_{i_{q_i}}) \\
\beta_{\text{pa}(j)} = (a_{j_1}, \ldots, a_{j_{r_j}}, y_{j_1}, \ldots, y_{j_{s_j}})
$$

(17)

denote the conditioning conjectures of $\text{pa } (X_i)$ and $\text{pa } (Y_j)$, respectively, where the indices for the
volitional states are defined separately from the indices for the stochastic states. For example,
consider the stochastic social model corresponding to the network

$$
\begin{array}{c}
X_1 \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
Y \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
X_2
\end{array}
$$

(18)

where the common syntax allows combining the volitional entities and the stochastic entity according
to the Bayesian network structure. The stochastic social model is of the form

$$
v_{X_1, X_2, Y}(a_1, a_2, y) = v_{X_1}(a_1)v_{X_2, Y}(a_2|a_1, y)p_{Y|X_1}(y|a_1).
$$

(19)

**Definition 2.** A stochastic conditional game, denoted $\{X, Y, \mathcal{A} \times \mathcal{Y}, \mathcal{V}, \mathcal{P}\}$, comprises a set
of volitional-stochastic states $\{X, Y\}$, a corresponding product action-realization set $\mathcal{A} \times \mathcal{Y}$, a set
of conditional payoff mass functions

$$
\mathcal{V} = \{v_{i|\text{pa}(i)}(|\alpha_{\text{pa}(i)}): A_i \rightarrow [0, 1] \forall \alpha_{\text{pa}(i)} \in A_{i_1} \times \cdots \times A_{i_{p_i}} \times Y_{i_1} \times \cdots \times Y_{i_{q_i}}, i = 1, \ldots, n\},
$$

(20)

and a set of conditional probability mass functions

$$
\mathcal{P} = \{p_{j|\text{pa}(j)}(|\beta_{\text{pa}(j)}): Y_j \rightarrow [0, 1] \forall \beta_{\text{pa}(j)} \in A_{j_1} \times \cdots \times A_{j_{r_j}} \times Y_{j_1} \times \cdots \times Y_{j_{s_j}}, j = 1, \ldots, m\}.
$$

(21)
Due to the common mathematical structure of stochastic states and volitional states, both types of states may be incorporated into a stochastic network comprising a set of stochastic state vertices (random variables) and volitional state vertices (agents). The edges that terminate at volitional states are conditional payoff mass functions and the edges that terminate at stochastic states are conditional probability mass functions. The fundamental theorem of Bayesian networks may then be applied to create a **stochastic social model** of the form

\[
v_{1:n,m}(a_1, \ldots, a_n, y_1, \ldots, y_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} v_{i|pa(i)}(a_i|\alpha_{pa(i)})p_{j|pa(j)}(y_j|\beta_{pa(j)}) .
\]  

A stochastic coordination function \( w_{1:n,m} \) can then be defined as

\[
w_{1:n,m}(a_{11}, \ldots, a_{nn}, y_1, \ldots, y_m) = \sum_{-a_{11} \cdots -a_{nn} -y_1 \cdots -y_m} v_{1:n,m}([a_{11}, \ldots, a_{1n}], \ldots, (a_{nn}, \ldots, a_{nn}), y_1, \ldots, y_m) .
\]  

For the example depicted by (18), the stochastic coordination function is

\[
w_{X_1X_2Y}(a_{11}, a_{22}, y) = \sum_{a_{21}, a_{12}} (v_{X_1X_2Y}((a_{11}, a_{12}), (a_{21}, a_{22}), y) .
\]  

For a dissociated stochastic network, the stochastic social model collapses to become stochastic coordination function, yielding

\[
w_{1:n}(a_{11}, \ldots, a_{nn}, y_1, \ldots, y_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} v_{i|pa(i)}(a_{ii})p_{j|pa(j)}(y_j|\beta_{pa(j)}) ,
\]  

where

\[
\alpha_{pa(i)} = (a_{i1i_1}, \ldots, a_{ipi_i}, y_{i1}, \ldots, y_{iq_i}) \quad \beta_{pa(j)} = (a_{j1j_1}, \ldots, a_{jjj_j}, y_{j1}, \ldots, y_{js_j}) .
\]  

A stochastic payoff for \( X_i \) is the \( i \)-th marginal of stochastic coordination, yielding yielding

\[
w_i(a_{ii}, y_1, \ldots, y_m) = \sum_{-a_{ii} -y_1 \cdots -y_m} w_{1:n,m}(a_{11}, \ldots, a_{nn}, y_1, \ldots, y_m) .
\]  

### 4.2 Expected Coordination

Given a stochastic conditional game \( \{X, Y, A \times \mathcal{Y}, \mathcal{V}, \mathcal{P}\} \), stochastic coordination is a multivariate mass function and therefore may be manipulated according to the rules of probability theory; thus, \( w_{1:n,m}(a_{11}, \ldots, a_{nn}, y_1, \ldots, y_m) \) may be expressed as

\[
w_{1:n,m}(a_{11}, \ldots, a_{nn}, y_1, \ldots, y_m) = w_{1:n|1:m}(a_{11}, \ldots, a_{nn}|y_1, \ldots, y_m)p_{1:m}(y_1, \ldots, y_m) ,
\]  

where

\[
p_{1:m}(y_1, \ldots, y_m) = \sum_{a_{11} \cdots a_{nn}} w_{1:n,m}(a_{11}, \ldots, a_{nn}, y_1, \ldots, y_m)
\]  

and

\[
w_{1:n|1:m}(a_{11}, \ldots, a_{nn}|y_1, \ldots, y_m) = \frac{w_{1:n,m}(a_{11}, \ldots, a_{nn}, y_1, \ldots, y_m)}{p_{1:m}(y_1, \ldots, y_m)}
\]
is the coordinatability function. This function is a measure of the degree of coordination given the specific random state $Y = y$.

Viewed as a function of the random vector $Y = \{Y_1, \ldots, Y_m\}$, coordinatability function is a multivariate random variable $w_{1:n|1:m}(a_{11}, \ldots, a_{nn}|Y_1, \ldots, Y_m)$ with expected value, termed expected coordinatability

$$
\overline{w}_{1:n}(a_{11}, \ldots, a_{nn}) = \mathbb{E}\{w_{1:n|1:m}(a_{11}, \ldots, a_{nn}|Y_1, \ldots, Y_m)\}
$$

(31)

Thus, by appealing to (28), expected coordinatability is the marginal of the stochastic coordination function with respect to the coordination conjecture profile $a = (a_{11}, \ldots, a_{nn})$:

$$
\overline{w}_{1:n}(a_{11}, \ldots, a_{nn}) = \sum_{y_1 \cdots y_m} w_{1:n|1:m}(a_{11}, \ldots, a_{nn}, y_1, \ldots, y_m).
$$

(32)

Also, the expected payoff for $X_i$ is

$$
\overline{w}_i(a_{ii}) = \sum_{y_1 \cdots y_m} w_i(a_{ii}, y_1, \ldots, y_m),
$$

(33)

where $w_i$ is given by (27).

4.3 Expected Utility and Risk on Stochastic Networks

In a traditional monetary context, utility serves as a measure of the satisfaction one receives as a function of one’s wealth. Utility theory can also play a role in the evaluation of stochastic influence networks. In this context, there are two manifestations of utility: one at the individual level with regard to performance, and one at the group level with regard to coordination. At the individual level, expected utility admits the conventional interpretations in terms of personal valuations such as material benefit and satisfaction. The concept of utility at the group level, are not naturally expressed in terms of “group benefit” or “group satisfaction,” since those terms may not be easily defined. The notion of social capital, however, may be an appropriate interpretation for a large class of social networks.

As discussed previously, coordinatability is an operational measure of joint intentionality, or the intrinsic ability of a collective to engage in systematic behavior, given the random states specified by the valuations $Y = y$. Let $U_{1:n}$ denote a convex strictly increasing von Neumann-Morgenstern utility. The coordination utility of the coordination profile $a = (a_{11}, \ldots, a_{nn})$ is

$$
u_{1:n|1:m}(a|y) = U_{1:n}[w_{1:n|1:m}(a|y)],
$$

(34)

and the expected coordinatability utility is

$$
\overline{\nu}_{1:n}(a) = \sum_y u_{1:n|1:m}(a|y)p_{1:m}(y).
$$

(35)

The coordination certainty equivalent, denoted $w_{1:nCE}(a)$, is the value whose utility is equal to the expected utility of the coordination conjecture profile, that is,

$$
U_{1:n}[w_{1:nCE}(a)] = \overline{\nu}_{1:n}(a),
$$

(36)
or, since $U_{1,n}$ is invertible,
\[ w_{1,n,C\!E}(A) = U_{1,n}^{-1}[\pi_{1,n}(A)]. \] (37)

The coordination certainty equivalent is the degree of joint intentionality such that its coordination utility is equal to the expected coordination utility. It is the amount of social capital which, if possessed with certainty, would yield the same utility as the expected utility of the outcome $a$.

The **coordination risk premium**, defined as
\[ R_{1,n}(a) = \pi_{1,n}(a) - w_{1,n,C\!E}(a), \] (38)
is the maximum amount of social capital the network is prepared to spend to avoid the outcome $a$.

Let $U_i$ denote a utility for $X_i$. The expected utility of $w_{i|1,m}(a_{ii}, Y)$, is
\[ \pi_i(a_{ii}) = \sum_{y} U_i[w_{i|1,m}(a_{ii}|y)]p_{1,m}(y). \] (39)

Given the expected utility $\pi_i$, the certainty equivalent is
\[ w_{i,C\!E} - U_i^{-1}[\pi_i(a_{ii})] \] (40)
and the risk premium is
\[ R_i(a_{ii}) = \pi_i(a_{ii}) = w_{i,C\!E}. \] (41)

## 5 Illustrative Scenario

As an illustration of a stochastic network, consider the following scenario. Three principals are considering the prospect of combining their resources to build a factory to manufacture widgets. One principal is charged with choosing between two locations, another is charged with choosing from among a set of four possible manufacturing processes, and the third must choose from among four varieties of widgets to manufacture. The decisions, however, must be made in an environment of political, environmental, and economic risk. The states and alternatives are defined as follows.

**Political influence**: $Y_1$ is the political environment that affects the desirability of location: $\mathcal{Y}_1 = \{H_1, H_2\}$.

**Location**: $X_1$ must choose between two locations as influenced by the political environment: $\mathcal{A}_1 = \{L_1, L_2\}$.

**Ecological impact**: $Y_2$ is the ecological environment that affects the process as a function of the location and which influences the manufacturing process: $\mathcal{Y}_2 = \{V_1, V_2\}$.

**Manufacturing Process**: $X_2$ must choose from among four processes as a function of the location, and which influences the profitability of the product: $\mathcal{A}_2 = \{P_1, P_2, P_3, P_4\}$.

**Economic state**: $Y_3$ is the economic state that affects the profitability of the product: $\mathcal{Y}_3 = \{E_1, E_2\}$.

**Manufactured Product**: $X_3$ must choose from among four widgets as a function of the manufacturing process: $\mathcal{A}_3 = \{W_1, W_2, W_3, W_4\}$.
The expected coordinatability is

\[ v_{X_1|Y_1} \]

where

- \( p_{Y_1} \) is the probability mass function governing \( Y_1 \).
- \( p_{Y_3} \) is the probability mass function governing \( Y_3 \).
- \( v_{X_1|Y_1} \) is the conditional payoff for \( X_1 \) given \( Y_1 \).
- \( p_{Y_2|X_1} \) is the conditional probability of \( Y_2 \) given \( X_1 \).
- \( v_{X_2|Y_2} \) is the conditional payoff for \( X_2 \) given \( Y_2 \).
- \( v_{X_3|X_2Y_3} \) is the conditional payoff for \( X_3 \) given \( Y_3 \) and \( X_2 \).

Thus, there are 256 possible states to consider when choosing from among the 32 possible alternatives.

The stochastic coordination function is

\[ w_{X_1X_2X_3Y_1Y_2Y_3}(a_{11}, a_{22}, a_{33}, y_1, y_2, y_3) = v_{X_1|Y_1}(a_{11}|y_1)v_{X_2|Y_2}(a_{22}|y_2)v_{X_3|X_2Y_3}(a_{33}|a_{22}, y_3)p_{Y_1}(y_1)p_{Y_2|X_1}(y_2|a_{11})p_{Y_3}(y_3). \] (43)

The coordinatability function is

\[ w_{X_1X_2X_3|Y_1Y_2Y_3}(a_{11}, a_{22}, a_{33}|y_1, y_2, y_3) = \frac{w_{X_1X_2X_3Y_1Y_2Y_3}(a_{11}, a_{22}, a_{33}, y_1, y_2, y_3)}{p_{Y_1Y_2Y_3}(y_1, y_2, y_3)} \] (44)

where

\[ p_{Y_1Y_2Y_3}(y_1, y_2, y_3) = \sum_{a_{11}a_{22}a_{33}} w_{X_1X_2X_3|Y_1Y_2Y_3}(a_{11}, a_{22}, a_{33}, y_1, y_2, y_3)p_{Y_1Y_2Y_3}(y_1, y_2, y_3). \] (45)

The expected coordinatability is

\[ \bar{w}_{X_1X_2X_3}(a_{11}, a_{22}, a_{33}) = \sum_{y_1y_2y_3} w_{X_1X_2X_3|Y_1Y_2Y_3}(a_{11}, a_{22}, a_{33}, y_1, y_2, y_3)p_{Y_1Y_2Y_3}(y_1, y_2, y_3). \] (46)

The expected payoffs for \( X_1 \), \( X_2 \), and \( X_3 \) are given by

\[ \bar{w}_{X_1}(a_{11}) = \sum_{a_{22}a_{33}} \bar{w}_{X_1X_2X_3}(a_{11}, a_{22}, a_{33}), \] (47)

\[ \bar{w}_{X_2}(a_{22}) = \sum_{a_{11}a_{33}} \bar{w}_{X_1X_2X_3}(a_{11}, a_{22}, a_{33}), \] (48)

and

\[ \bar{w}_{X_3}(a_{33}) = \sum_{a_{11}a_{22}} \bar{w}_{X_1X_2X_3}(a_{11}, a_{22}, a_{33}). \] (49)
for
\[(a_{11}, a_{22}, a_{33}) \in \{L_1, L_2\} \times \{P_1, P_2, P_3, P_4\} \times \{W_1, W_2, W_3, W_4\}.\] (50)

Let \(U\) denote a concave strictly increasing utility function. The coordinatability utility function is
\[u_{x_1,x_2,x_3|y_1,y_2,y_3}(a_{11}, a_{22}, a_{33}|y_1, y_2, y_3) = U[w_{x_1,x_2,x_3|y_1,y_2,y_3}(a_{11}, a_{22}, a_{33}|y_1, y_2, y_3)]\] (51)
from which the expected coordination utility is
\[\bar{u}_{x_1,x_2,x_3}(a_{11}, a_{22}, a_{33}) = \sum_{y_1,y_2,y_3} u_{x_1,x_2,x_3|y_1,y_2,y_3}(a_{11}, a_{22}, a_{33}|y_1, y_2, y_3)p_{y_1,y_2,y_3}(y_1, y_2, y_3).\] (52)

6 Conclusions

Relying on conventional noncooperative game theory as a model of an influence network is problematic, since that theory is designed for preference concepts governed by the rationality principle of narrow self-interest, and is therefore susceptible to congruence violations when the socially motivated solution concepts are employed.

Expanding the preference concept to coordinated self-interest requires that categorial payoffs be replaced by conditional payoffs that enable agents to respond to influence. The result is the development of a natural aggregation mechanism that requires the conditional payoffs to conform to the syntax of probability theory, therefore guaranteeing social coherence.

The requirement that conditional payoffs conform to the probability axioms enables stochastic states, who are governed by conditional probabilities, to be directly embedded into the network, thereby enabling the application of Bayesian network theory to generate a comprehensive social model that may be used to create coordinated solution concepts. Group behavior is expressed via a stochastic coordination function that leads to generation of expected coordination and expected payoffs for the volitional states. Finally, expected utility theory may be applied, thereby enabling an assessment of certainty equivalence and risk primia for the group and for its members.
References


