Random Walks in Dividend Yields and Bubbles *

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Abstract

A nonstationary dividend yield, having a unit root, is seen as proof of bubbles (Craine 1993). This inference is not valid. A sufficient condition for the absence, respectively presence of bubbles is the uniform divergence, respectively uniform convergence of the dividend yield series. I use this criterion to show that a random walk dividend yield must be bubble-free if a positive deterministic trend or a large positive drift is present. I also construct an example where the equilibrium dividend yield is a random walk without a deterministic trend or drift, but bubbles are still absent.

Keywords: bubbles, random walk dividend yield, unit root tests

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1 Introduction

Testing for bubbles, defined as the price of an asset in excess of the discounted present value of dividends, requires assumptions on the stochastic discount factor (*SDF* henceforth), which is unobservable. The early tests of bubbles (for a survey, see Gurkaynak 2008) assumed a constant SDF, which contradicts a large amount of evidence on returns predictability and time-varying risk premium (Campbell, Lo, and MacKinlay 1997, Chapter 2).

Craine (1993) proposed an elegant way to bypass the SDF misspecification problem, predicated only on the stationarity of the SDF. A stationary SDF and a stationary dividend growth result in a stationary dividend yield if bubbles are absent. Therefore nonstationarity of the dividend yield, or prices that are more explosive than dividends, is interpreted as evidence of bubbles, while stationarity of the dividend yield is seen as proof of no bubbles.

Evans (1991) shows, through simulations, that the presence of periodically collapsing bubbles is virtually undetectable by standard unit root and cointegration tests. Such bubbles can be made in fact stationary and lead to a stationary dividend yield, as shown in Bidian (2014c). Therefore *stationary* dividend yields do *not* guarantee the *absence* of bubbles.

This paper shows that *nonstationary* dividend yields do *not* guarantee the *existence* of bubbles either. Even when the (log) dividend yield has a unit root (follows a random walk), what matters for the existence of bubbles, first and foremost, is the presence of (deterministic) time trends. A positive time trend guarantees the absence of bubbles, despite the presence of a unit root. A negative time trend ensures that bubbles are present, even if the dividend yield is trend-stationary.

These results are demonstrated without any assumptions on the stochastic discount factor, using a criterion involving the convergence of the dividend yield series. Montrucchio (2004) proves that if the dividend yield series *diverges uniformly*, then bubbles cannot exist. I show that the converse also holds: if the dividend yield series *converges uniformly*, then bubbles must exist.

In the absence of a time trend but with a non-zero drift, the criterion can still be applied if the error terms of the random walk are small relative to the size of the drift. In this case, a positive (negative) drift ensures the absence (presence) of bubbles. Without a drift or with a small drift relative to the size of the errors, the uniform convergence or divergence of the dividend yield cannot be established. In this situation, it is unclear whether the nonstationarity of the dividend yield guarantees that bubbles exist. I show that this is not the case, by constructing an economy with a bubble-free equilibrium in which the dividend yield follows a random walk with zero or positive drift (and no deterministic time trend).

In the example economy, the SDF is nonstationary, while dividend growth is stationary. Craine (1993) argues that the SDF is "at least mean stationary", due to indirect evidence such as the apparent stationarity of returns. However, even mean stationarity of the SDF is questionable. For instance, Hall, Anderson, and Granger (1992) report the inability to reject unit roots in interest rates. Moreover, assuming stationary returns when one tests for bubbles is tantamount to rule out most bubbles, and represents a circular reasoning. Indeed, the returns on a bubbly asset are a weighted average of the fundamental (bubble-free) returns and the rate of growth of the bubble component. The weight of the fundamental return in the bubbly return is given by the ratio of the fundamental to the bubbly price. Therefore a bubble with nonstationary rate of growth or with a nonstationary ratio of the fundamental to the bubble-inflated price would create nonstationary returns.

Craine (1993) cannot reject the null hypothesis of a unit root in the annual and quarterly log dividend yield process for the NYSE index for the period 1927-1989, and for the annual S&P composite index for the period 1872-1988. He concludes that bubbles are present in (some of) the constituents of those indexes, but points out that the apparent unit root in the log dividend yield might actually reflect the existence of a low frequency component in the SDF. Another issue of unit root tests is that they have low power against different alternatives. In fact, Koustas and Serletis (2005), using exactly the same data set as Craine (1993), reject the hypothesis of a unit root in the log dividend yield in favor of stationary fractionally integrated processes. Similarly, Diba and Grossman (1988) find that prices and dividends for the annual S&P composite index for the period 1871-1986 are cointegrated and interpret this as proof of absence of bubbles.

By contrast, I show that even if all the econometric problems can be tackled and the dividend yield is guaranteed to be a random walk, bubbles do not have to exist. Moreover, I point out that stationarity-based tests do not pay attention to deterministic time trends or drifts in the dividend yield process, which have a first-order effect on the existence of bubbles, and trump the presence of a unit root.

2 Convergence of the dividend yield series and bubbles

Time periods are indexed by the set $\mathbb{N} := \{0, 1, \ldots\}$. The uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ and by the filtration $(\mathcal{F}_t)_{t=0}^{\infty}$, which is an increasing sequence of σ-algebras on the set of states of the world Ω. Each σ-algebra $\mathcal{F}_t$ is interpreted as the information available at date t. The conditional expectation given $\mathcal{F}_t$ (with respect to the probability P) is denoted by $E_t(\cdot)$, with $E_0(\cdot)$ being simply written as $E(\cdot)$. The notation $(x_t)_{t=0}^{\infty}$ (or just (x_t)) represents a *process*, that is a sequence of random variables such that $x_t : \Omega \to \mathbb{R}$ is $\mathcal{F}_t$-measurable for all $t \in \mathbb{N}$. All equalities and inequalities (implicitly) hold only P-almost surely (*a.s.*).

Consider an asset with dividends (d_t) and strictly positive ex-dividend prices (p_t). The "fundamental theorem of asset pricing", which follows from the absence of arbitrage opportunities, ensures under mild conditions the existence of a strictly positive *state price density* (a_t) that *martingale-prices* all the assets:

$$a_t p_t = E_t a_{t+1}(p_{t+1} + d_{t+1}), \forall t \geq 0. \tag{2.1}$$

In particular, martingale-pricing holds when agents are subject to wealth restrictions in the form of debt (respectively borrowing) constraints, requiring the beginning (respectively end) of period value of their portfolio to exceed some specified bounds (Bidian 2014a). Iterating in (2.1),

$$p_t = \underbrace{\frac{1}{a_t} E_t \left(\sum_{s=t+1}^{\infty} a_s d_s \right)}_{f_t} + \underbrace{\frac{1}{a_t} \lim_{s\to\infty} E_t a_s p_s}_{b_t}. \tag{2.2}$$

The term f_t is the *fundamental value* of the asset at t computed as the present value of dividends discounted by (a_s). The term b_t represents the price of the asset in excess of its fundamental value, and it is interpreted as a *bubble* at period t. By (2.1), the process $(a_t b_t)$ is a martingale. Hence prices (p_t) are free of bubbles with respect to the state price density (a_t) ($b_t = 0$ for all t) if and only if $b_0 = 0$, or equivalently,

$$\lim_{t\to\infty} E a_t p_t = 0. \tag{2.3}$$

Let $R_{t+1} := (p_{t+1} + d_{t+1})/p_t$, respectively $R^f_{t+1} := (f_{t+1} + d_{t+1})/f_t$ be the actual return, respectively the "fundamental" return of the asset from t to $t+1$. The returns on an asset containing a bubble are a weighted average between the fundamental returns and the bubble rate of growth,

$$R_{t+1} = \frac{f_t}{p_t} \cdot R^f_{t+1} + \left(1 - \frac{f_t}{p_t}\right) \frac{b_{t+1}}{b_t}. \tag{2.4}$$

Bubbles with nonstationary growth or bubbles that lead to a nonstationary ratio of the fundamental to the actual price will result in general in nonstationary returns.

The empirical literature testing for bubbles focuses on the stationarity of the dividend yield, rather than the stationarity of returns. The dividend yield in the absence of bubbles is stationary, as long as the SDF is stationary and the dividend growth is stationary. This is explained in detail in Craine (1993), and it is due to the fact that stationarity is preserved by measurable transformations of a process (Kallenberg 2002, Lemma 10.1). The failure to reject the existence of a unit root in the dividend yield is interpreted as evidence of bubbles. Stationarity of the dividend yield is viewed as evidence of no bubbles.

In economies with more severe portfolio constraints, such as short sale constraints (rather than wealth constraints), martingale-pricing might not hold. However, the assets are nevertheless *supermartingale-priced* (Bidian 2014a), thus there exists a strictly positive process (a_t) such that for any asset (with prices (p_t) and dividends (d_t)),

$$a_t p_t \geq E_t a_{t+1}(p_{t+1} + d_{t+1}), \forall t \geq 0. \tag{2.5}$$

Let $g_t := p_t - E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})$ and $m_t = \frac{1}{a_t} E_t \sum_{s \geq t} a_s g_s$, for all $t \geq 0$. Iteration in (2.5) shows that the asset price admits the decomposition $p_t = f_t + m_t + b_t$. The additional component m_t can be interpreted as the *resale option* or the *convenience yield* from holding the asset at t (Bidian 2014a). The results of this paper hold under supermartingale-pricing, rather than just martingale-pricing.

The behavior of the dividend yield process is crucially related to bubbles, but rather than its nonstationarity (stationarity), it is the uniform divergence (convergence) of the dividend yield series that is a sufficient condition for the absence (presence) of bubbles. This is true without any assumptions imposed on the unobservable SDF. For all $t \geq 0$, let

$$\xi_t := \prod_{s=0}^{t} \left(1 + \frac{d_s}{p_s}\right) \text{ and } \zeta_t := \sum_{s=0}^{t} \frac{d_s}{p_s}. \tag{2.6}$$

Since $\exp(\zeta_t) \geq \xi_t \geq \zeta_t$, the convergence (divergence) of the *dividend yield series* (ζ_t) is equivalent to the convergence (divergence) of (ξ_t).

The series (ζ_t) *diverges uniformly* if

$$\forall\, N \in \mathbb{N}, \exists T_N \in \mathbb{N} \text{ such that } \zeta_t \geq N, \forall t \geq T_N. \tag{2.7}$$

Similarly, the series (ζ_t) *converges uniformly* is (ζ_t) is bounded from above by a real number.

Proposition 2.1. *If the dividend yield series diverges uniformly, then there are no bubbles.*

Proof. Using (2.5) and (2.6),

$$a_t p_t \xi_t \geq E_t a_{t+1} p_{t+1} \xi_{t+1}. \tag{2.8}$$

Thus $(a_t p_t \xi_t)$ is a (positive) supermartingale. Since (ζ_t) diverges uniformly, it follows that (ξ_t) diverges uniformly. Therefore for all $N \in \mathbb{N}$, there exists $T_N \in \mathbb{N}$ such that

$$a_0 p_0 \xi_0 \geq E(a_t p_t \xi_t) \geq N \cdot E(a_t p_t), \forall t \geq T_N.$$

It follows that $E(a_t p_t) \to 0$. $\square$

Proposition 2.1 was obtained by Montrucchio (2004, Theorem 2) for the martingale-pricing case. Unfortunately the uniform divergence condition is quite strong. Clearly, if the dividend yield process is bounded from below, that is if there exists a $\nu > 0$ such that

$$\frac{d_t}{p_t} \geq \nu, \forall t \geq 0, \tag{2.9}$$

then the uniform boundedness condition is trivially satisfied. However it is unclear how to relax the uniform divergence assumption or condition (2.9) further. Bidian (2011) gives an example where (2.9) holds *infinitely often*[1] but bubbles exist. In that example the dividend yield series diverges a.s. but not uniformly.

Proposition 2.1 admits a converse.

Proposition 2.2. *If the dividend yield series converges uniformly, then bubbles must be present under any state price density that martingale-prices the assets.*

Proof. Assume that there exists $N \in \mathbb{R}$ such that $\zeta_t \leq N$, for all t. It follows that

[1]This means that $P(\cap_{n \geq 0} \cup_{t \geq n} d_t/p_t \geq \nu) = 1$.

$\xi_t \le e^N$, for all t. Therefore if (a_t) martingale-prices the assets,

$$E(a_t p_t) \ge E(a_t p_t \xi_t \cdot \exp(-N)) = a_0 p_0 \xi_0 \cdot \exp(-N) > 0, \forall t,$$

which shows that a bubble must be present. $\square$

3 Random walk dividend yield

I show that bubbles do not have to exist, even when econometricians identify without any doubt the presence of a unit root in the dividend yield. Indeed, assume that the dividend yield follows a random walk with (possibly) drift and deterministic trend:

$$\ln \frac{d_{t+1}}{p_{t+1}} = \alpha + \delta(t+1) + \ln \frac{d_t}{p_t} + \nu_{t+1}, \forall t \ge 0. \tag{3.1}$$

The error terms (ν_t) are assumed to be i.i.d. with zero mean and support $[\underline{\nu}, \bar{\nu}]$, for some $\underline{\nu} < 0 < \bar{\nu}$. It follows that

$$\frac{d_t}{p_t} = \frac{d_0}{p_0} \exp\left(\alpha t + \delta t(t+1)/2) + \sum_{n=1}^{t} \nu_n\right), \forall t \ge 1, \tag{3.2}$$

and therefore

$$\frac{d_0}{p_0} \sum_{s=0}^{t} \exp(\alpha s + \delta s(s+1)/2 + \underline{\nu} s) \le \sum_{s=0}^{t} \frac{d_s}{p_s} \le \frac{d_0}{p_0} \sum_{s=0}^{t} \exp(\alpha s + \delta s(s+1)/2 + \bar{\nu} s). \tag{3.3}$$

The inequalities in (3.3) can be used to obtain sufficient conditions for the uniform convergence or divergence of the dividend yield series.

3.1 Deterministic time trend

I consider first the case when deterministic time trends are present ($\delta \neq 0$).

If $\delta > 0$, (3.3) shows that the dividend yield series diverges uniformly and therefore bubbles *cannot exist* (Proposition 2.1), despite the dividend yield process being a *nonstationary* process and having a stochastic trend.

If $\delta < 0$, notice that $\exp(\alpha s+\delta s(s+1)/2+\bar{\nu}s) < \exp(\delta s)$, for all $s > (\alpha+\bar{s})/(-\delta)$. The dividend yield series converges uniformly by (3.3). Therefore with a negative deterministic trend, bubbles must *exist* (Proposition 2.2) if the assets are martingale-priced. This remains true even in the absence of a stochastic trend, that is, for *trend stationary processes.* Indeed, assume for example that $\ln \frac{d_t}{p_t} = \alpha + \delta t + \nu_t$ for all t, and $\delta < 0$. The dividend yield is *trend stationary.* Nevertheless the dividend yield series converges uniformly and bubbles must be present, by Proposition 2.2:

$$\sum_{t=0}^{\infty} \frac{d_t}{p_t} \leq \exp(\alpha + \bar{\nu}) \sum_{s=0}^{\infty} \exp(\delta t) = \frac{\exp(\alpha + \bar{\nu})}{1 - \exp(\delta)}.$$

3.2 Drift but no deterministic time trend

Assume now that there is no deterministic time trend ($\delta = 0$) but there is a drift ($\alpha \neq 0$). By the law of large numbers and (3.2), $\lim_{t\to\infty} \left(\frac{d_t}{p_t}\right)^{1/t} = \exp(\alpha)$ a.s. If $\alpha < 0$, respectively $\alpha > 0$, Cauchy's root test implies that the dividend yield series $\sum \frac{d_t}{p_t}$ converges, respectively diverges, a.s. Moreover, by (3.3) with $\delta = 0$, the dividend yield series converges (respectively diverges) uniformly if $\alpha > -\underline{\nu} > 0$ (respectively $\alpha < -\bar{\nu} < 0$). Therefore with small errors relative to the size of the drift one can conclude that a positive drift rules out bubbles, while a negative drift and martingale-pricing guarantees the presence of bubbles.

However, with arbitrarily large error terms (ν_t), a non-zero drift guarantees just the a.s. convergence or divergence of the dividend yield series. This divergence (convergence) was not proved to be uniform, therefore bubbles cannot be ruled out (shown to exist) using this argument.

3.3 No drift or deterministic time trend

When $\alpha = 0$ and $\delta = 0$, $\limsup_{t\to\infty} \sum_{s=1}^{t} \nu_t = \infty$ (Kallenberg 2002, Proposition 9.14) and therefore

$$\limsup_{t\to\infty} \sum_{s=1}^{t} \frac{d_t}{p_t} \geq \limsup_{t\to\infty} \frac{d_t}{p_t} = \frac{d_0}{p_0} \limsup_{t\to\infty} \exp\left(\sum_{n=1}^{s} \nu_s\right) \geq \frac{d_0}{p_0}(1 + \limsup_{t\to\infty} \sum_{n=1}^{s} \nu_s) = \infty.$$

The dividend yield series diverges a.s., but the divergence might not be uniform.

4 Example

Without a deterministic trend or without a large drift relative to the size of errors, the issue whether bubbles must exist due to the presence of a stochastic trend is left unresolved by the previous analysis.

In what follows, I construct an economy where the equilibrium log dividend yield follows a random walk without deterministic trend, and with a drift that can be zero or any positive real number. In this example, there are no bubbles, despite the nonstationarity of the dividend yield.

There is a continuum of agents. Each agent has preferences $E\sum_{t\geq 0} \beta^t u(c_t^i)$, where $\beta \in (0,1)$ and $u(c) = -c^{-1}$. There is only one asset, with dividends (d_t) with stationary growth ((d_{t+1}/d_t) stationary). Dividends are chosen such that $(1/d_t)$ is uniformly integrable and $E\frac{1}{d_t} \to 0$. The aggregate endowment is (c_t), with $c_t = \gamma_t d_t$, where $\gamma_t \geq 1$. Notice that the utility of the "representative" agent is finite, since

$$E\sum_{t\geq 0}\beta^t \frac{1}{c_t} \leq E\sum_{t\geq 0}\beta^t \frac{1}{d_t} = \sum_{t\geq 0}\beta^t E\frac{1}{d_t} < \infty.$$

Let (ν_t) be a sequence of i.i.d. zero-mean shocks, independent of (d_t). Construct the asset prices (p_t) such that

$$\ln\frac{d_{t+1}}{p_{t+1}} = \alpha + \ln\frac{d_t}{p_t} + \nu_{t+1}, \forall t \geq 0,$$

where d_0/p_0 is normalized to one and $\alpha \geq 0$. Equivalently, $\frac{d_t}{p_t} = e^{\alpha t + z_t}$, where $z_0 := 0$ and $z_t := \sum_{n=1}^{t}\nu_n$ for $t \geq 1$. I assume that the aggregate endowment grows sufficiently fast:

$$\left(\frac{\gamma_{t+1}}{\gamma_t}\right)^2 \geq \beta\frac{d_t}{d_{t+1}} \cdot \frac{1 + e^{\alpha(t+1)+z_{t+1}}}{e^{\alpha+\nu_{t+1}}}. \tag{4.1}$$

The SDF is taken to be

$$\frac{a_{t+1}}{a_t} := \frac{p_t}{p_{t+1}} \cdot \frac{1}{1 + d_{t+1}/p_{t+1}} = \frac{p_t}{p_{t+1}} \cdot \frac{\xi_t}{\xi_{t+1}} = \frac{d_t}{d_{t+1}} \cdot \frac{\exp(\alpha + \nu_{t+1})}{1 + \exp(\alpha(t+1) + z_{t+1})}. \tag{4.2}$$

By construction, the fundamental valuation equation (2.1) is satisfied and $(a_t p_t)$ is a decreasing sequence. By the monotone convergence theorem,

$$Ea_t p_t \to 0 \Leftrightarrow a_t p_t \to 0 \text{ a.s.} \Leftrightarrow \xi_t \to \infty \text{ a.s. } \Leftrightarrow \zeta_t \to \infty \text{ a.s..}$$

From Sections 3.2 and 3.3, it is known that (ξ_t) and (ζ_t) diverge a.s. since $\alpha \geq 0$ (see (2.6) for the definition of $(\xi_t), (\zeta_t)$). Therefore there are no bubbles in the asset under this SDF.

I show next that the posited dividend yield and SDF can be sustained in a no-trade equilibrium, where agents have symmetric holdings of the asset. The construction follows Constantinides and Duffie (1996). I need to verify their conditions (5) and (6), which are

$$Ea_t \to 0, \quad \frac{a_{t+1}}{a_t} \geq \beta \left(\frac{c_{t+1}}{c_t} \right)^{-2}. \tag{4.3}$$

Using (4.2) and (4.1), (4.3) is indeed satisfied:

$$\begin{aligned} Ea_t &= E(\exp(\alpha t + z_t) d_t^{-1} \xi_t^{-1}) < E d_t^{-1} \to 0, \\ \frac{a_{t+1}}{a_t} = \frac{d_t}{d_{t+1}} \frac{\exp(\alpha + \nu_{t+1})}{1 + \exp(\alpha(t+1) + z_{t+1})} &\geq \beta \frac{\gamma_t^2}{\gamma_{t+1}^2} \frac{d_t^2}{d_{t+1}^2} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-2}. \end{aligned}$$

Equations (8)-(10) in Constantinides and Duffie (1996) show how to construct agents' individual endowments to guarantee that there exists an equilibrium with no trade that supports the price process (p_t) as an equilibrium (their Proposition 1).

5 Conclusion

A nonstationary (stationary) dividend yield is equated to the presence (absence) of bubbles in the literature testing for bubbles. I show that the presence of unit roots in the dividend yield, and hence its nonstationarity, does not guarantee the existence of a bubble. On the other hand, stationarity of the dividend yield does not guarantee the absence of bubbles (Evans 1991, Bidian 2014c). These two facts combined paint a bleak picture of stationarity-based tests for bubbles.

It is the uniform convergence (respectively divergence) of the dividend yield series that ensures the presence (respectively absence) of bubbles, rather than the stationarity of the dividend yield. When the log dividend yield is nonstationary and follows a random walk, a positive deterministic time trend or a large positive drift relative to the size of errors dwarfs the presence of the stochastic trend (unit root) and ensures that bubbles cannot exist, as the dividend yield series diverges uniformly. Even in the absence of a deterministic trend or a large (relative to errors) positive drift, bubbles can be absent. To show this, I construct an economy where the equilibrium dividend yield follows a random walk without trend and with zero or positive drift.

Using theoretical insights to rule out or to identify bubbles can be more fruitful rather than persisting in using testing procedures that are only tangential to bubbles. For example, the uniform divergence of the dividend yield series criterion shows that bubbles cannot exist in an environment where firms have to pay at least a fixed fraction (no matter how small) of their stock price as dividends. On the other hand, now we know that in environments with limited enforcement of contracts, bubbles are ubiquitous (Hellwig and Lorenzoni 2009, Bidian 2011, Bidian 2014b).

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