# Stationary Bubbles *

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**Abstract**

Tests of asset price bubbles typically focus on the stationarity properties of the dividend yield. Evidence of nonstationarity in the dividend yield is viewed as proof of bubbles, while stationarity is interpreted as absence of bubbles. For economies with arbitrary pricing kernels but stationary risk-free rates, I show that there exist periodically collapsing bubbles of the type introduced by Evans (1991) that are strictly stationary. Such bubbles give rise to stationary dividend yields.

**Keywords:** bubbles, stationary dividend yield, unit root tests

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## 1 Introduction

A *bubble* is defined as the price of an asset in excess of its *fundamental value*, which is the discounted present value of dividends. Bubbles are martingales in discounted

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terms, and therefore grow on average at the rate of interest rates.

Testing for bubbles requires assumptions on the *stochastic discount factor* (*SDF* henceforth), which is unobservable. Gurkaynak (2008) provides a survey of the literature on bubble tests. The early tests assumed a constant SDF, which contradicts a large amount of evidence on returns predictability and time-varying risk premia (Campbell, Lo, and MacKinlay 1997, Chapter 2).

The more recent tests of bubbles avoid the potential SDF misspecification problem and focus on the stationarity properties of the dividend yield. This methodology is predicated on the stationarity of the (unobservable) SDF and of the dividend growth (which seems to be the case in the data), which implies stationary dividend yields if bubbles are absent (Craine 1993). Nonstationarity of the dividend yield, or equivalently, prices that are more explosive (less stationary) than dividends, is interpreted as evidence of bubbles, while stationarity of the dividend yield is seen as proof of the absence of bubbles.

I show that this inference is not valid. Stationarity of the dividend yield does not rule out stationary bubbles. For an arbitrary SDF that gives rise to stationary risk-free rates, I construct a class of *strictly stationary* bubbles that periodically collapse, as in Evans (1991). A strictly stationary process (random sequence) has a distribution that is invariant under shifts (time-translation). Such a strictly stationary bubble results in a strictly stationary and *covariance stationary*[1] dividend yield (even if the bubble itself is not covariance stationary).

Evans (1991) shows, through Monte Carlo simulations, that the presence of periodically collapsing bubbles in economies with *constant SDF* is virtually undetected

[1]A random sequence $(x_t)_{t=0}^{\infty}$ is covariance stationary if $cov(x_t, x_{t+s}) = cov(x_n, x_{n+s})$, for all $n, t, s$. Any strictly stationary process $(x_t)$ is also covariance stationary if the first and second moments of $x_t$ (for all $t$) are finite.

by standard unit root and cointegration tests. The results of my paper give a theoretical justification for this finding, by showing that such periodically collapsing bubbles, in addition to being *conditionally stationary*,[2] are in fact strictly stationary. Moreover, such strictly stationary bubbles can be constructed even in economies with *arbitrary* SDF, as long as the risk-free rates are stationary. Concretely, I prove the existence of a stationary distribution for these bubbles, via a fixed point argument. Moreover, in the particular case of a constant SDF (the case analyzed by Evans (1991)), this stationary distribution can be constructed in explicit form.

The results of this paper cast doubt on stationarity-based empirical tests of bubbles. Not surprisingly, there are a bewildering number of contradictory findings. Craine (1993) analyzes the existence of a unit root in the annual and quarterly log dividend yield process for the NYSE, and for the annual S&P composite index. He cannot reject the null hypothesis of a unit root and concludes that bubbles. In contrast, Diba and Grossman (1988), respectively Koustas and Serletis (2005), find that prices and dividends for the annual S&P composite index are integrated of order one, respectively fractionally integrated, and interpret this as proof of absence of bubbles. The same fractional integration is found by Cunado, Gil-Alana, and de Gracia (2005) in the NASDAQ index at daily and weekly frequencies, but not at monthly frequencies.

There are Markov regime switching tests designed specifically to detect the periodically collapsing bubbles of Evans (1991), reviewed in Gurkaynak (2008, Sections 3.3 and 3.4). They are sensitive to the way of modeling the switching probabilities, and can lead to contradictory findings even when applied to the same S&P500 data set. Additionally, since these tests assume that the bubble can switch between two

[2]That is, the conditional distributions of the process are invariant to time shifts.

states, but fundamentals do not change, they cannot distinguish between regime switching fundamentals rather than collapsing bubbles. Moreover, the type of collapsing bubbles they try to detect are likely to form a tiny subset in the class of all stationary bubbles.

# 2 Stationary bubbles

Time periods are indexed by the set $\mathbb{N} := \{0, 1, \ldots\}$. The uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ and by the filtration $(\mathcal{F}_t)_{t=0}^{\infty}$, which is an increasing sequence of $\sigma$-algebras on the set of states of the world $\Omega$. Each $\sigma$-algebra $\mathcal{F}_t$ is interpreted as the information available at date $t$. The conditional expectation given the period $t$ information $\mathcal{F}_t$ (with respect to the probability $P$) is denoted by $E_t(\cdot)$, with $E_0(\cdot)$ being written as $E(\cdot)$. For $A \in \mathcal{F}$, $\mathbf{1}_A$ is the indicator function of the set $A$, defined as $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if $x \notin A$.

Consider an asset that pays dividends given by the random sequence ("process" henceforth) $(d_t)_{t=0}^{\infty}$ (for each $t \geq 0$, $d_t$ is $\mathcal{F}_t$-measurable) and trades at (ex-dividend) prices $(p_t)_{t=0}^{\infty}$. By the "fundamental theorem of asset pricing", which follows from the absence of arbitrage opportunities in general environments, there exists a strictly positive *pricing kernel* $(a_t)$ that *martingale-prices* all the assets:

$$a_t p_t = E_t a_{t+1}(p_{t+1} + d_{t+1}), \forall t \geq 0. \tag{2.1}$$

The SDF $(m_{t+1})_{t \geq 0}$ is defined by $m_{t+1} := a_{t+1}/a_t$.

By iteration in (2.1),

$$p_0 = \underbrace{\frac{1}{a_0} E \sum_{t=1}^{\infty} a_t d_t}_{f_0} + \underbrace{\frac{1}{a_0} \lim_{t\to\infty} E a_t p_t}_{b_0} . \tag{2.2}$$

The term $f_0$ represents the *fundamental value* of the asset computed as the present value of dividends discounted by $(a_t)$. The term $b_t$ represents a *bubble* at period 0, under the pricing kernel $(a_t)$. Therefore prices $(p_t)$ are free of bubbles under the pricing kernel $(a_t)$ if

$$\lim_{t\to\infty} E a_t p_t = 0. \tag{2.3}$$

I construct a class of *strictly stationary* periodically collapsing bubbles of the type introduced by Evans (1991), but associated to an arbitrary SDF. It is only assumed that the SDF $(m_{t+1})$ gives rise to strictly stationary risk-free rates $(R_t)$, where $R_t := (E_t m_{t+1})^{-1}$. Let $\mathcal{R}$ and $F^R$ be the support and cumulative distribution function of $R_t$. A bubble is a nonnegative process $(\varepsilon_t)$ such that $\varepsilon_t = E_t m_{t+1} \varepsilon_{t+1}$, for all $t$. I assume that $\mathcal{R} = [1+\underline{r}, 1+\bar{r}] \subset (1,\infty)$. This assumption is not essential, as explained in footnote 4.

Let $(v_{t+1})_{t\geq 0}$ be a sequence of iid random variables with a cumulative distribution function $F^v$, support $[b_0, b_1] \subset (0,\infty)$ and mean $\delta$. Let $(\eta_{t+1})_{t\geq 0}$ be iid, with $\eta_{t+1}$ taking the value 1 with probability $\pi \in (0,1)$ and 0 with probability $1-\pi$. The sequences $(v_{t+1})_{t\geq 0}$ and $(\eta_{t+1})_{t\geq 0}$ are independent of each other and of the SDF $(m_{t+1})$ (depend only on extrinsic uncertainty).[3]

[3] In fact, all that is required is that at each period $t$, $\eta_{t+1}$ and $v_{t+1}$ are uncorrelated with $m_{t+1}$ conditional on the information available at $t$.

Define the process $(\varepsilon_t)$ by

$$\varepsilon_{t+1} = \upsilon_{t+1}(1-\eta_{t+1}) + \eta_{t+1} f(R_t\varepsilon_t), \forall t \geq 0, \tag{2.4}$$

where $\varepsilon_0 \in [b_0, \infty)$ is arbitrary and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and given by

$$f(x) := \pi^{-1}\left(x - (1-\pi)\delta\right). \tag{2.5}$$

The process $(\varepsilon_t)$ collapses to the interval $[b_0, b_1] \subset (0, \infty)$ with probability $1-\pi$, while with probability $\pi$ it keeps growing. Parameters $\delta, \underline{r}, \pi$ and $b_0$ are chosen such that $(1+\underline{r})b_0 \geq \pi b_0 + (1-\pi)\delta$, which guarantees that $f(R_t\varepsilon_t) \geq b_0$ and therefore the process $(\varepsilon_t)$ is positive.[4] Moreover, $(\varepsilon_t)$ is a bubble for the SDF $(m_t)$, since

$$E_t(m_{t+1}\varepsilon_{t+1}) = E_t(m_{t+1})E_t(\varepsilon_{t+1}) = \frac{1}{R_t}\left((1-\pi)\delta + \pi f(R_t\varepsilon_t)\right) = \varepsilon_t.$$

The distribution of $\varepsilon_{t+1}$ conditional on the information available at $t$ is

$$F^{\varepsilon}_{\varepsilon_{t+1}|\varepsilon_t, R_t} := (1-\pi)F^{\upsilon} + \pi H_{f(R_t\varepsilon_t)},$$

where $H_{f(R_t\varepsilon_t)}(x) := \mathbf{1}_{f(R_t\varepsilon_t)\geq x}$ is the Heaviside step function at $f(R_t\varepsilon_t)$. Thus the (vector) process $(\varepsilon_t, R_t)$ is *conditionally stationary* (the conditional distributions are

[4] The assumption that $1+\underline{r} > 1$ is used here, to guarantee positivity of $(\varepsilon_t)$. It can be dispensed with by assuming, as in Evans (1991), that the bubble has a chance to collapse only after exceeding some threshold $\alpha > 0$, and simply grows at the risk free rate while smaller than $\alpha$:

$$\varepsilon_{t+1} = R_t \cdot \mathbf{1}\varepsilon_t \leq \alpha + (\upsilon_{t+1}(1-\eta_{t+1}) + \eta_{t+1} f(R_t\varepsilon_t)) \cdot \mathbf{1}\varepsilon_t > \alpha, \forall t \geq 0.$$

Choosing $\alpha > (1-\pi)\delta/(1+\underline{r})$ guarantees that $f(R_t\varepsilon_t) > 0$ whenever $\varepsilon_t > \alpha$ and therefore the process $(\varepsilon_t)$ is positive. The existence of the invariant distribution of $(\varepsilon_t)$ via a fixed point argument follows in an identical way. What is lost, in terms of tractability, is the analytical expression for the invariant distribution when the risk free rates are constant.

invariant to time shifts). In what follows, I show that $\varepsilon$ can be made in fact *strictly stationary*. Strict stationarity implies conditional stationarity, but the converse is not true.

Denote by $F_t^\varepsilon$ the (unconditional) cumulative distribution function of $\varepsilon_t$, for $t \geq 0$. Notice that

$$F_{t+1}^\varepsilon(x) = E(\mathbf{1}_{\varepsilon_{t+1}\leq x}) = E\left(E(\mathbf{1}_{\varepsilon_{t+1}\leq x}|\varepsilon_t, R_t)\right) =$$
$$= E\left((1-\pi)F^v(x) + \pi\mathbf{1}_{f(R_t\varepsilon_t)\leq x}\right).$$

Thus

$$F_{t+1}^\varepsilon(x) = (1-\pi)F^v(x) + \pi\int_{\mathcal{R}} F_t^\varepsilon(f^{-1}(x)/r)dF^R(r). \tag{2.6}$$

Equation (2.6) defines an operator $T$ on the metric space of distributions on the real line endowed with the Kolmogorov (uniform) metric, with $F_{t+1}^\varepsilon = TF_t^\varepsilon$. For two cumulative distribution functions $G_1, G_2$, the Kolmogorov uniform distance is $d_K(G_1, G_2) = \sup_{x\in\mathbb{R}} |G_1(x) - G_2(x)|$. Notice that $d_K(TG_1, TG_2) \leq \pi \cdot d_K(G_1, G_2)$, thus $T$ is a contraction. Therefore $T$ has a fixed point $F^\varepsilon$, which represents the invariant distribution of $(\varepsilon_t)$. It follows that $(\varepsilon_t)$ is strictly stationary if the initial $\varepsilon_0$ is drawn from the distribution $F^\varepsilon$.

When the risk-free rates are constant ($R_t = R$ for all $t$), there is an analytic expression for the stationary distribution $F^\varepsilon$. Let $\hat{f}(x) := f(Rx)$, with $f$ given by (2.5). By (2.6), $F_{t+1}^\varepsilon = (1-\pi)F^v + \pi F_t^\varepsilon \circ \hat{f}^{-1}$. Iterating in this formula,

$$F_n^\varepsilon = \sum_{t=0}^{n-1}(1-\pi)\pi^t F^v \circ \hat{f}^{-t} + \pi^n F_0^\varepsilon \circ \hat{f}^{-n}, \forall n \geq 1.$$

Hence if $\varepsilon_0$ is drawn from the "stationary" distribution

$$F_0^\varepsilon = F^\varepsilon := \sum_{t=0}^{\infty}(1-\pi)\pi^t F^v \circ \hat{f}^{-t},$$

then $\{\varepsilon_t\}_{t=0}^{\infty}$ is strictly stationary.

The stationary bubbles constructed here lead to stationary dividend yields if injected in asset prices. Clearly if $(p_t, d_t, \varepsilon_t)$ is strictly stationary, then $(d_t/(p_t + \varepsilon_t))$ is strictly stationary, since a measurable transformation preserves stationarity (Kallenberg 2002, Lemma 10.1). Moreover, if $(d_t/p_t)$ is covariance stationary (has finite first and second order moments), then $(d_t/(p_t+\varepsilon_t))$ is also covariance stationary, since $d_t/(p_t + \varepsilon_t) \leq d_t/p_t$.

Due to dividend growth, $(d_t)$ and $(p_t)$ are not stationary in general. However, the dividend growth seems stationary in the data. If $(d_{t+1}/d_t, m_{t+1})$ is stationary, then $(d_t/p_t)$ is stationary (Craine 1993), and also $(m'_{t+1})$ defined by $m'_{t+1} := m_{t+1}d_{t+1}/d_t$ is stationary. Using the construction in this paper, there exists a strictly stationary process $(\varepsilon'_t)$ such that $\varepsilon'_t = E_t m'_{t+1}\varepsilon'_{t+1}$, for all $t$. Then clearly $(\varepsilon_t)$ given by $\varepsilon_t := \varepsilon'_t d_t$ is a bubble associated to $(m_t)$ in that $\varepsilon_t = E_t m_{t+1}\varepsilon_{t+1}$. Furthermore, $d_t/(p_t + \varepsilon_t) = 1/(p_t/d_t + \varepsilon'_t)$. Therefore the bubble process $(\varepsilon_t)$ injected in the asset prices $(p_t)$ preserves the strict stationarity of the dividend yield.

# 3 Conclusion

Concluding that bubbles are absent based on the stationarity of the dividend yield (confirmed through some testing procedure) is not warranted, even if the premise of a stationary SDF is accepted. In fact, there exist strictly stationary bubbles in economies with arbitrary SDF, as long as the risk-free rates are stationary. Such bubbles collapse periodically as in Evans (1991), but do not rely on his assumption of a constant SDF. Bidian (2011) shows how to introduce bubbles in asset prices by a tightening of agents' debt limits, in economies with arbitrary market structures.

A strictly stationary bubble preserves the stationary of the dividend yield. This provides a theoretical justification of why Evans's (1991) Monte Carlo simulations indicate that periodically collapsing bubbles are virtually undetectable by stationarity-based tests. It also gives an insight into why such tests often lead to conflicting findings.

## References


Bidian, F. (2011): "Essays on Asset Price Bubbles," Ph.D. thesis, University of Minnesota.

Campbell, J. Y., A. Lo, and C. MacKinlay (1997): *The Econometrics of Financial Markets*. Princeton University Press, Princeton.

Craine, R. (1993): "Rational Bubbles: A Test," *Journal of Economic Dynamics and Control*, 17(5-6), 829–846.

Cunado, J., L. A. Gil-Alana, and F. P. de Gracia (2005): "A Test for Rational Bubbles in the NASDAQ Stock Index: A Fractionally Integrated Approach," *Journal of Banking and Finance*, 29(10), 2633–2654.

Diba, B., and H. Grossman (1988): "Explosive Rational Bubbles in Stock Prices?," *American Economic Review*, 78(3), 520–530.

Evans, G. W. (1991): "Pitfalls in Testing for Explosive Bubbles in Asset Prices," *American Economic Review*, 81(4), 922–30.

Gurkaynak, R. S. (2008): "Econometric tests of asset price bubbles: taking stock," *Journal of Economic Surveys*, 22(1), 166–186.

Kallenberg, O. (2002): *Foundations of modern probability*. Springer, 2nd edn.

Koustas, Z., and A. Serletis (2005): "Rational Bubbles or Persistent Deviations from Market Fundamentals?," *Journal of Banking and Finance*, 29(10), 2523–2539.